Exercise Sheet 1

Saddle-point problems, and why naive finite elements fail

Mixed finite element methods — a crash course

20 July 2026

Setting and conventions. Notation as in the lecture. Tags: [T] pen and paper; [N] notebook, starting from `L1A_1d_suite.ipynb` or `L1B_q1p0_cavity.ipynb`; the notebook cells tagged `S1-S4` contain material omitted from the lecture, on which this sheet relies. Starred parts are optional. Several derivations were deliberately left to this sheet: the solvability theory for the alternating mode (Problem 1), and the $-\frac{1}{4}$ and $h^2/60$ identities together with the limit $u_h \to u/6$ (Problem 2). Throughout, the one-dimensional mixed Laplacian reads: find $\sigma \in V_h$, $u \in Q_h$ with

$$(\sigma, \tau) + (\tau', u) = 0 \quad \forall \tau \in V_h, \qquad (\sigma', v) = -(f, v) \quad \forall v \in Q_h,$$

on a uniform mesh of $(0,1)$ with n elements.

Problem 1 [T] — The one-dimensional alternating mode

Let $V_h = Q_h =$ continuous P_1.

(a) Show that $\{\tau_h' : \tau_h \in V_h\}$ is the space of *all* piecewise constants, and deduce that $q_h \in \ker B^T$ if and only if $\int_K q_h = 0$ on every element. Conclude that $\ker B^T = \mathrm{span}\{\chi_h\}$, where χ_h has nodal values $\chi_i = (-1)^i$ — for every n.

(b) Show that the block matrix is singular, and that if (σ_h, u_h) solves the discrete system, then so does $(\sigma_h,\, u_h + t\chi_h)$ for every $t \in \mathbb{R}$. (Even when a solution exists, the method expresses no preference within a one-parameter family.)

(c) Show that the discrete system is solvable if and only if $(f, \chi_h) = 0$. Prove that $\int_0^1 \chi_h = 0$ for every n, and that χ_h is symmetric about $x = \frac{1}{2}$ when n is even and antisymmetric when n is odd. Deduce: $f = 1$ is compatible for every n; $f = \pi^2 \sin(\pi x)$ is compatible if and only if n is odd; $f(x) = x$ is compatible if and only if n is even. (Compare with self-study cell `S1`. Solvability depends on the parity of the number of elements.)

(d) Run self-study cell `S2` and explain both outputs by means of (b) and (c): why is the regularized solution apparently correct for $f = 1$, and an alternating mode of amplitude $\sim 10^5$ for $f = \pi^2 \sin(\pi x)$, at $n = 64$ — and why does the latter become apparently correct again at $n = 65$? State in one sentence what the apparently correct result demonstrates about the method.

(e) Keep $Q_h =$ continuous P_1 and enrich the flux space to $V_h =$ continuous P_2. Show that $\ker B^T = \{0\}$. State in one sentence whether this establishes that the enriched pair is a satisfactory method.

Problem 2 [T + N] — An invertible pair that does not converge

Let $V_h =$ continuous P_2, $Q_h =$ discontinuous P_0.

(a) Show that the block matrix is invertible for every n.

(b) Show that $K_h = \mathrm{span}\{1\} \oplus \mathrm{span}\{b_K\}_K$, with b_K the quadratic bubble ($b = 4t(1-t)$ on the reference element). Compute $\|b_K\|_0^2 = \frac{8h}{15}$ and $\|b_K'\|_0^2 = \frac{16}{3h}$, and conclude that

$$\alpha_h := \min_{\tau_h \in K_h} \frac{\|\tau_h\|_0^2}{\|\tau_h\|_0^2 + \|\tau_h'\|_0^2} \;\le\; \frac{h^2}{10}\big(1 + O(h^2)\big).$$

(c*) Sharpen the bound: on each element take $\tau_h = \beta\,(b_K - \frac{2}{3})$ (the bubble minus its mean; the term $-\frac{2}{3}\beta$ is realized by the single global constant when β is the same on all elements). Show that the ratio improves to $\frac{h^2}{60}(1 + O(h^2))$. The measured value in self-study cell `S4` is $\alpha_h/h^2 = 0.0167$ on every mesh: the bound is sharp.

(d) Derive the $-\frac{1}{4}$ identity stated in the lecture: the constraint fixes the vertex increments of σ_h exactly, and testing with $\tau_h \in K_h$ gives $(\sigma_h, \tau_h) = 0$. Writing $\sigma_h|_K = \ell_K + \beta_K b_K$, deduce $\beta_K = -\frac{5}{4}\,\ell_K(m_K)$ and hence $\sigma_h(m_K) = -\frac{1}{4}\,\ell_K(m_K)$. For $f = 1$ conclude that, although the exact flux $\sigma = \frac{1}{2} - x$ belongs to V_h, the relative L^2 flux error tends to $\sqrt{5/6} \approx 0.913$. [N] Verify these results numerically; self-study cell `S3` contains the overlay of u_h, u and $u/6$.

(e*) Explain the limit $u_h \to u/6$: testing with the P_1 hat function at the vertex x_i (admissible, since $P_1 \subset P_2$) gives $u_h|_{K_{i+1}} - u_h|_{K_i} = (\sigma_h, \varphi_i)$. Show that, to leading order, $(\sigma_h, \varphi_i) = \frac{h}{6}\,\sigma(x_i) + O(h^2)$, i.e. u_h increases with one sixth of the true slope.

Problem 3 [N] — Local conservation and discontinuous coefficients

Consider $-(\kappa u')' = f$ on $(0,1)$, $u(0) = u(1) = 0$, with $f = 1$ and $\kappa = 1$ on $(0, \frac{1}{2})$, $\kappa = 100$ on $(\frac{1}{2}, 1)$. The physical flux is $q = -\kappa u' = -\sigma$.

(a) [T, preliminary] Compute the exact solution: $\sigma(x) = C - x$ with $C = \frac{103}{404}$; the flux q is continuous across the interface, while u' jumps by a factor 100.

(b) Solve with (i) the primal method (continuous P_1 for u, flux recovered as $q_h^{\mathrm{pr}} := -\kappa u_h'$) and (ii) the mixed method (P_1–P_0 with $a(\sigma, \tau) = (\kappa^{-1}\sigma, \tau)$, flux $q_h := -\sigma_h$). Use an *aligned* mesh ($n = 32$: interface on a node) and a *misaligned* one ($n = 33$: interface inside an element). Plot the three fluxes (q, q_h^{pr}, q_h) near the interface in all cases. Describe the behaviour of q_h^{pr} within the cut element at $n = 33$, and that of q_h.

(c) For each element, tabulate the balance residual $r_K := \left|\int_K f - \left(q_h(x_K^-) - q_h(x_{K-1}^+)\right)\right|$ and the nodal flux jumps $[q_h](x_i)$, for both methods, and report the maxima. (Expected outcome: for the mixed method both quantities vanish to machine precision — the local-conservation property stated in the lecture, proved in general in Lecture 3; for the primal method, $r_K = O(h)$ on every element and an $O(1)$ flux jump at the interface in the misaligned case.)

(d) Discuss in one paragraph which of the observed advantages of the mixed method are structural, and persist in higher dimensions, and which are specific to one dimension. (The lecture identified one such one-dimensional effect.)

Problem 4 [T + N] — Spurious modes in two dimensions

(a) [T] On a uniform $N \times N$ quadrilateral mesh of the unit square, consider velocities in continuous Q_1 (vanishing on the whole boundary), pressures in P_0, and $b(\boldsymbol{v}, q) = -(\operatorname{div} \boldsymbol{v}, q)$. Let q_{cb} denote the checkerboard mode (± 1 according to the colouring of the cells). Show that $q_{\mathrm{cb}} \in \ker B^T$. *Hint:* it suffices to test with $\boldsymbol{v} = \varphi_j \boldsymbol{e}_1$ and $\boldsymbol{v} = \varphi_j \boldsymbol{e}_2$ for the interior nodal hat functions φ_j; compute $\int_K \partial_x \varphi_j = \pm h/2$ on each of the four cells adjacent to the node, and sum with the checkerboard signs.

(b) [N] Assemble the cavity system of `L1B` for $N = 4, 6, 8, 12, 16$; compute all singular values; count the numerical zeros and plot the pressure parts of the null vectors. Report the count as a function of N: are the constant and the checkerboard always present, and do additional modes ever appear?

(c) [N] The triangular analogue: on a triangular mesh of the unit square, take P_1–P_1 for Stokes (continuous P_1 velocities and pressures) and the same regularized cavity. (i) Solve (with the ε-shift and the constant-pressure null space handled as in the notebook) and plot p_h; describe the result. (ii) Using the provided helper, assemble K_V (the velocity stiffness matrix with boundary conditions), M_Q and B, and compute the full spectrum of the generalized eigenvalue problem

$$\mathsf{B}\mathsf{K}_V^{-1}\mathsf{B}^T\mathbf{q} = \lambda\, \mathsf{M}_Q\mathbf{q}.$$

Count the numerically zero eigenvalues (relative tolerance 10^{-10}), identify the constant mode among them, and report the count. (iii*) Let $\tilde{Q}_h$ denote the orthogonal complement of the zero modes found in (ii). Compute $\tilde{\beta}_h := \sqrt{\lambda_{\min}}$ on $\tilde{Q}_h$ for a sequence of refinements and tabulate it against h. State in one sentence which quantity the theory of Lecture 2 must therefore control.