Mixed finite element methods — a crash course

Exercise sheet 2: the Babuška–Brezzi theory

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Conventions. Problems are tagged [T] (theory) or [N] (notebook); starred parts are optional. In one dimension $H(\mathrm{div}) = H^1$; the V-norm is $\|\cdot\|_{\mathrm{div}}$ and the Q-norm is $\|\cdot\|_0$; for Stokes the velocity norm is $\|\nabla\cdot\|_0$. The notebook `L2_constants.ipynb` and the scripts `verify_beta_1d.py`, `verify_p1p1_tri.py` are part of the distributed bundle; numbered results refer to the Lecture 2 slides.

Problem 1 [T] — matrix form of the stability constants

Let $\{\varphi_i\}_{i=1}^N$ and $\{\psi_j\}_{j=1}^M$ be bases of V_h and Q_h, let G_V and G_Q be the Gram matrices of the V- and Q-inner products, and set $\mathsf{A}_{ik} = a(\varphi_k, \varphi_i)$, $\mathsf{B}_{ji} = b(\varphi_i, \psi_j)$.

(a) Prove that $\beta_h = \sigma_{\min}\big(\mathsf{G}_V^{-1/2}\mathsf{B}^T\mathsf{G}_Q^{-1/2}\big)$ and that β_h^2 is the smallest eigenvalue of $\mathsf{B}\,\mathsf{G}_V^{-1}\mathsf{B}^T\mathbf{q} = \lambda\,\mathsf{G}_Q\mathbf{q}$. Prove the analogous characterization $\alpha_h = \lambda_{\min}$ of $\mathsf{Z}^T\mathsf{A}\mathsf{Z}\,\mathbf{w} = \mu\,\mathsf{Z}^T\mathsf{G}_V\mathsf{Z}\,\mathbf{w}$, where the columns of Z form a basis of $\ker\mathsf{B}$.

(b) For the one-dimensional P_1–P_1 pair with $n = 2$ elements, write the 3×3 matrix B and determine $\ker\mathsf{B}^T$ by hand; conclude that $\beta_h = 0$.

(c) Prove the four monotonicity statements: enlarging V_h (with Q_h fixed) does not decrease β_h and does not increase α_h; enlarging Q_h (with V_h fixed) does not increase β_h and does not decrease α_h. Verify all four statements against the measured table of the lecture (notebook cell 4).

Problem 2 [T] — a Fortin operator for the working pair

Let $\Omega = (0,1)$, let V_h be the continuous P_1 space and $Q_h = P_0$ on a uniform mesh (the pair of Lecture 1), and let $I_h : H^1(0,1) \to V_h$ denote vertex interpolation, which is well defined by the embedding $H^1(0,1) \hookrightarrow C[0,1]$.

(a) Show that $\int_K (I_h\tau)'\,dx = \int_K \tau'\,dx$ on every element K, and conclude that (F1) holds.

(b) Show that $\|(I_h\tau)'\|_0 \le \|\tau'\|_0$.

(c) Show that $\|I_h\tau\|_0 \le \sqrt{2}\,\|\tau\|_{H^1}$, with a constant independent of h.

(d) Conclude from Theorem 11 that $\beta_h \ge \beta/\sqrt{3}$. Compare with the direct bound $\beta_h \ge \sqrt{2/3}$ obtained from the antiderivative lifting, and explain why the direct argument does not extend to $d \ge 2$, while the operator I_h possesses a generalization that is presented in Lecture 3. Combine the uniform bound with $\alpha_h = 1$ (Observation 9), Corollary 8, and Theorem 7 to prove that $\|\sigma - \sigma_h\|_{\mathrm{div}} = O(h)$ for this pair, and state explicitly which two observations of Lecture 1 — the order-two L^2 flux rate and the nodal exactness — are one-dimensional effects that Theorem 7 does not imply.

(e*) Compute the continuous constant exactly. Show that $\beta^2 = \inf_q (Tq, q)_0/\|q\|_0^2$ with $Tq := t'_q$, where t_q is the Riesz representative of $b(\cdot, q)$ in V; derive the strong form $-t'' + t = -q'$ with the natural conditions $(t' - q)(0) = (t' - q)(1) = 0$; eliminate $q = t'/\lambda$; conclude that the eigenvalues are $\lambda_k = k^2\pi^2/(1 + k^2\pi^2)$, $k \geq 1$, with eigenfunctions $q \propto \sin(k\pi x)$, and hence $\beta = \pi/\sqrt{1 + \pi^2} \approx 0.9529$. Compare with the measured limit of β_h (notebook cell 5).

Problem 3 [T] — the converse of Fortin's criterion

Assume $\beta_h \geq \beta_* > 0$ for the family (V_h, Q_h).

(a) For $v \in V$ define $G_v \in Q'_h$ by $G_v(q_h) := b(v, q_h)$; show that $\|G_v\|_{Q'_h} \leq \|b\| \, \|v\|_V$.

(b) Apply Lemma 2(iii)–(iv) to the pair (V_h, Q_h) to obtain $\Pi_h v \in K_h^\perp \cap V_h$ with $b(\Pi_h v, q_h) = b(v, q_h)$ for all $q_h \in Q_h$ and $\|\Pi_h v\|_V \leq \|G_v\|_{Q'_h}/\beta_h$.

(c) Show that $v \mapsto \Pi_h v$ is linear.

(d) Conclude (F1)–(F2) with $C_\Pi \leq \|b\|/\beta_*$, and state the resulting equivalence between uniform inf-sup stability and the existence of a uniformly bounded Fortin family.

(e) Show directly that no operator satisfying (F1) exists for the one-dimensional P_1–P_1 pair. Hint: test (F1) with $q_h = \chi_h$.

Problem 4 [N] — measuring the theory

This problem uses `L2_constants.ipynb` (cells 4–5, 7–8, S1, S2) and the two verification scripts.

(a) Reproduce the constants table at $n = 128$; confirm the three exact entries to solver precision and the ratio $\alpha_h/h^2 = 1/60$ to five digits; verify the four monotonicity statements of Problem 1(c); observe the convergence of β_h from above to the value of Problem 2(e*).

(b) For the triangular P_1–P_1 pair, report the number of numerically zero eigenvalues on `UnitSquareMesh(N, N)` for $N \in \{8, 16, 32\}$ and identify the constant mode among them; compute the essential constant $\widetilde{\beta}_h$ and its decay rate. Explain in at most one paragraph why any nonzero kernel already excludes uniform stability (Theorem 10), and what the decay of $\widetilde{\beta}_h$ adds to the diagnosis.

(c*) Worst-case data. With $w_h \in K_h$ a single interior bubble of the P_2–P_0 pair, set $F := a(\cdot, w_h)$ and $G := 0$; verify that the discrete solution is $(w_h, 0)$; measure $C_h := \|\sigma_h\|_{\mathrm{div}}/\|F\|_{V'_h}$ under refinement and compare its growth with the bounds $\alpha_h^{-1/2} \leq C_h \lesssim \alpha_h^{-1}$; repeat with the alternating bubble field and explain which datum attains the order of the upper bound, and why.

(d**) Open exploration: the exact modes. Using the perturbation and pattern options of `verify_p1p1_tri.py`, reproduce the counts 8 (structured), 5 (randomly perturbed), and 7 (alternating pattern, perturbed); visualize two structured modes; determine which modes survive perturbation. A classification of the surviving modes is not part of the course materials; document your observations.