Mixed finite element methods — a crash course

Exercise sheet 3: $H(\mathrm{div})$ and the mixed Laplacian

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Tags: [T] = theoretical exercise, [N] = notebook experiment (notebooks `L3A`, `L3B`). Parts marked with an asterisk are optional. Result numbers refer to the Lecture 3 slides (Lemma 15 – Proposition 29). Throughout, $\mathcal{T}_h$ is a shape-regular triangulation; for a triangle K with vertices $\boldsymbol{p}_0, \boldsymbol{p}_1, \boldsymbol{p}_2$, the edge e_k is the edge opposite $\boldsymbol{p}_k$, and $\boldsymbol{n}_k$ is the unit normal of e_k pointing out of K.

Problem 1 [T] — RT_0 element calculus

Let K be a triangle of area $|K|$ and define

$$\boldsymbol{\varphi}_k(\boldsymbol{x}) \;=\; \frac{\boldsymbol{x}-\boldsymbol{p}_k}{2\,|K|}, \qquad k = 0,1,2.$$

(a) Show that $\boldsymbol{\varphi}_k \in RT_0(K)$, that $\operatorname{div}\boldsymbol{\varphi}_k = 1/|K|$, and that the normal trace $\boldsymbol{\varphi}_k \cdot \boldsymbol{n}_j$ is constant on each edge e_j, with

$$\int_{e_j} \boldsymbol{\varphi}_k \cdot \boldsymbol{n}_j \, ds \;=\; \delta_{kj}, \qquad j,k \in \{0,1,2\}.$$

Conclude that $(\boldsymbol{\varphi}_k)_{k=0}^{2}$ is the basis of $RT_0(K)$ dual to the flux degrees of freedom $N_j(\boldsymbol{\tau}) = \int_{e_j} \boldsymbol{\tau}\cdot\boldsymbol{n}_j\,ds$. (Hint: $(\boldsymbol{x}-\boldsymbol{p}_k)\cdot\boldsymbol{n}_j$ is constant along e_j, and $\boldsymbol{p}_k \in e_j$ for $j \neq k$.)

(b) Re-derive unisolvence (Proposition 17 for $k = 0$) directly from (a): if all three degrees of freedom of $\boldsymbol{\tau} \in RT_0(K)$ vanish, then $\operatorname{div}\boldsymbol{\tau} = 0$, hence $\boldsymbol{\tau}$ is a constant vector, and two linearly independent normal directions force $\boldsymbol{\tau} = \boldsymbol{0}$.

(c) Let $\hat{K}$ be the reference triangle with vertices $(0,0)$, $(1,0)$, $(0,1)$ and let K be the triangle with vertices $(1,0)$, $(1,1)$, $(0,1)$. Compute the Piola transform (slide 10) of the reference basis function associated with the hypotenuse of $\hat{K}$ and verify Proposition 19(i): the edge flux of the transformed function over the corresponding edge of K equals the reference edge flux.

(d)* On the structured $N \times N$ triangulation of the unit square, mark the positions and directions of the global RT_0–P_0 degrees of freedom in a sketch, and compare with the staggered (Marker-and-Cell) arrangement of finite difference methods: normal velocities at face midpoints, scalars at cell centers.

Problem 2 — one dimension, and exactness by counting

(a) [T] On $\Omega = (0,1)$ with a uniform mesh, let V_h be the continuous P_1 space and $Q_h = P_0^{\mathrm{disc}}$. Show that the canonical interpolant Π_h^{RT} of Lecture 3 reduces to vertex interpolation I_h, and that the commuting property $(I_h\tau)' = P_h(\tau')$ is the statement that the derivative of the interpolant on each element equals the mean value of τ' there. Conclude that the Fortin operator constructed in Exercise 2 of Lecture 2 was the one-dimensional canonical interpolant, as announced there.

(b) [T] Conclude that in one dimension $\operatorname{div} V_h = Q_h$ and $K_h \subset K$, and identify K_h (the discrete kernel of the pair) as the one-dimensional space of constant functions.

(c)* [T] Let $\mathcal{T}_h$ be a triangulation of a simply connected polygon, with $\#V$ vertices, $\#E$ edges, and $\#T$ triangles. Prove exactness of the discrete complex $P_1^{\text{cont}} \xrightarrow{\textbf{curl}} RT_0 \xrightarrow{\text{div}} P_0^{\text{disc}}$ at the middle slot by counting:

(i) $\dim \textbf{curl}\big(P_1^{\text{cont}}(\mathcal{T}_h)\big) = \#V - 1$, and $\textbf{curl}\big(P_1^{\text{cont}}\big) \subseteq \ker\big(\text{div}\,|_{RT_0}\big)$;

(ii) $\dim\ker\big(\text{div}\,|_{RT_0}\big) = \#E - \#T$, using $\text{div}\, RT_0(\mathcal{T}_h) = P_0^{\text{disc}}(\mathcal{T}_h)$ (Corollary 22);

(iii) Euler's formula $\#V - \#E + \#T = 1$ closes the argument.

State, without proof, what changes on a domain with g holes.

Problem 3 — local conservation, elementwise and on patches

Data as in the notebooks: $u = \sin(\pi x)\sin(\pi y)$, $\kappa = 1$ unless stated otherwise.

(a) [N] Run cell 7 of `L3A`. For the P_1 Lagrange solution, prove the identity observed there: since ∇u_h is constant on each element, the net flux through every closed element boundary vanishes, and therefore the elementwise balance defect equals $\big|\int_K f\big|$ exactly on every element. For the RT_0–P_0 solution, cite the result that makes the defect vanish to machine precision (Proposition 24).

(b) [T] Show that the P_1 Lagrange solution satisfies a weaker, vertex-based balance: for every interior vertex z with hat function φ_z and patch $\omega_z = \operatorname{supp}\varphi_z$,

$$\int_{\omega_z} \nabla u_h \cdot \nabla\varphi_z \, dx \;=\; \int_{\omega_z} f\,\varphi_z \, dx.$$

Explain why no elementwise balance in the form of Proposition 24 can hold for this method: the numerical flux $-\nabla u_h$ has no single-valued normal component on interior edges.

(c) [N] Run cell 9 of `L3A` (layered coefficient, $\kappa \in \{1, 10^{-3}\}$). Report the elementwise balance defect and inspect the flux field across the interface. Compare with Exercise 3 of Lecture 1: identify which of the one-dimensional observations made there were structural (they persist here) and which were one-dimensional effects.

(d) [N] Run cell 13 of `L3A` (balances on random unions of elements). Prove the statement printed by the cell: for the mixed method, elementwise conservation and the single-valuedness of $\boldsymbol{\sigma}_h \cdot \boldsymbol{n}$ (Corollary 20) imply the exact balance $\int_{\partial\omega} \boldsymbol{q}_h \cdot \boldsymbol{n}\, ds = \int_\omega f\, dx$ on every union ω of elements, because the contributions of interior edges cancel in pairs.

Problem 4 — hybridization on two triangles; the exact inf-sup constant

Let $\Omega = (0,1)^2$ be split into two triangles K_1, K_2 by the diagonal from $(1,0)$ to $(0,1)$; take $\kappa = 1$, $f = 1$, $u_D = 0$. The hybridized RT_0–P_0 method (slide 27) uses broken fluxes (three degrees of freedom per triangle), one scalar per triangle, and one multiplier λ on the shared edge: nine unknowns.

(a) [T] Assemble the 9×9 hybrid system in the block form of slide 27. For the local 4×4 blocks $\begin{pmatrix} \mathsf{A}_K & \mathsf{s}_K \\ \mathsf{s}_K^T & 0 \end{pmatrix}$ (flux mass matrix and divergence column of one element), the midpoint quadrature rule is exact for the entries of A_K.

(b) [T] Show that each local block is invertible by applying Lemma 1 of Lecture 1 on the element: A_K is symmetric positive definite, and $\ker \mathsf{s}_K = \{0\}$ trivially since $\mathsf{s}_K \neq \mathbf{0}$.

(c) [T] Eliminate the local unknowns and reduce the system to a single equation $S\lambda = r$. Verify $S > 0$, in agreement with Proposition 28, and verify Theorem 27 on this example: the reconstructed flux has a single-valued normal component on the shared edge, so the pair $(\boldsymbol{\sigma}_h, u_h)$ coincides with the conforming mixed solution.

(d)* [T] Prove the formula for the continuous inf-sup constant of the pair $H(\mathrm{div};\Omega) \times L^2(\Omega)$ used on slide 24:

$$\beta = \sqrt{\frac{\mu_1}{1+\mu_1}},$$

where μ_1 is the first Dirichlet eigenvalue of $-\Delta$ on Ω. Proceed as in the one-dimensional case (Lecture 2, Exercise 2(e*)): the Riesz representative $\boldsymbol{t} \in H(\mathrm{div};\Omega)$ of the functional $\boldsymbol{\tau} \mapsto (\mathrm{div}\,\boldsymbol{\tau}, q)$ satisfies $(\boldsymbol{t}, \boldsymbol{\tau}) + (\mathrm{div}\,\boldsymbol{t}, \mathrm{div}\,\boldsymbol{\tau}) = (\mathrm{div}\,\boldsymbol{\tau}, q)$ for all $\boldsymbol{\tau}$; show that $\psi := \mathrm{div}\,\boldsymbol{t} - q$ belongs to $H_0^1(\Omega)$ and satisfies $-\Delta\psi + \psi = -q$, using the surjectivity of the normal trace (Theorem 16) to identify the boundary condition; expand in Dirichlet eigenfunctions. Evaluate the formula on $\Omega = (0,1)$ (recovering the constant of Lecture 2) and on the unit square, and compare with the values of β_h measured in cell 5 of `L3A`.