Mixed finite element methods — a crash course

**Exercise sheet 4: Stokes, computations, outlook**

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Tags: [T] = theoretical exercise, [N] = notebook experiment (notebooks `L4A`, `L4B`). Parts marked with an asterisk are optional. Result numbers refer to the Lecture 4 slides (Theorem 30 – Corollary 46). Throughout, $\mathcal{T}_h$ is a shape-regular triangulation; $\lambda_0, \lambda_1, \lambda_2$ are the barycentric coordinates of a triangle $K$; $\varphi_e$ denotes the quadratic edge basis function of an edge $e$ (value 1 at the midpoint of $e$, value 0 at every other $P_2$ node); the Stokes sign convention is $b(\boldsymbol{v}, q) = -(\operatorname{div} \boldsymbol{v}, q)$.

**Problem 1 [T] — Bubbles and the MINI element**

Let $K$ be a triangle of area $|K|$ and let $b_K = \lambda_0\lambda_1\lambda_2$ denote its cubic bubble. The MINI velocity space enriches the continuous $P_1$ space by one bubble per element and component (slide 14).

(a) Using the integration formula $\int_K \lambda_0^a\lambda_1^b\lambda_2^c\,dx = 2|K|\,a!\,b!\,c!/(a{+}b{+}c{+}2)!$, show that $\int_K b_K\,dx = |K|/60$ and that $\int_K \varphi_e\,dx = |K|/3$. (The first value calibrates the bubble correction in (b); the second is used in Problem 2.)

(b) Reproduce the proof of Lemma 37 with explicit constants. For $\boldsymbol{v} \in (H_0^1(\Omega))^2$ let $R_h\boldsymbol{v}$ be the quasi-interpolant of Lemma 33 and define, per element and component,

$$\Pi_h\boldsymbol{v}|_K = R_h\boldsymbol{v}|_K + \boldsymbol{\beta}_K\,b_K, \qquad \boldsymbol{\beta}_K = \Big(\int_K b_K\,dx\Big)^{-1}\int_K (\boldsymbol{v} - R_h\boldsymbol{v})\,dx.$$

Derive the local bound $\|\nabla(\boldsymbol{\beta}_K b_K)\|_{0,K} \le C_B\,h_K^{-1}\|\boldsymbol{v} - R_h\boldsymbol{v}\|_{0,K}$ from the scaling of $b_K$ under shape regularity, and assemble the global stability bound with the constant $C_R(1+C_B M^{1/2})$ stated on slide 16, identifying the patch-overlap number $M$.

(c)* Compute the three-dimensional value $\int_K \lambda_0\lambda_1\lambda_2\lambda_3\,dx = |K|/840$ on a tetrahedron (the formula reads $\int_K \lambda^{\boldsymbol{\alpha}}\,dx = 6|K|\,\boldsymbol{\alpha}!/(|\boldsymbol{\alpha}|+3)!$ there), and state how the constants of (b) change in three dimensions.

(d) Count the dimensions on a triangulation with $\#V$ vertices and $\#T$ triangles: $\dim V_h^{\mathrm{MINI}} = 2(\#V + \#T)$ and $\dim Q_h = \#V$. Verify that the counting inequality $\dim Q_h \le \dim V_h$ of Lecture 1 holds for MINI, and observe that it holds equally for the unstable pair $P_1$–$P_1$; conclude, once more, that the inequality is necessary and not sufficient.

**Problem 2 [T] — Hood–Taylor and the macroelement technique**

Let $M$ be a macroelement (a union of triangles of $\mathcal{T}_h$) and let

$$N_M = \{\, q \in P_1^{\mathrm{cont}}(M) \;:\; b(\boldsymbol{v}, q) = 0 \text{ for all } \boldsymbol{v} \in V_h \cap (H_0^1(M))^2 \,\}$$

be its local null space (slide 20), where $V_h$ is the continuous $P_2$ velocity space.

(a) For the macroelement of two triangles $K_1$, $K_2$ sharing an edge $e$, show that $V_h \cap (H_0^1(M))^2$ is spanned by $\varphi_e\,\boldsymbol{e}_1$ and $\varphi_e\,\boldsymbol{e}_2$, and derive from $b(\varphi_e\boldsymbol{a}, q) = 0$ the pair relation $|K_1|\,\nabla q|_{K_1} + |K_2|\,\nabla q|_{K_2} = \boldsymbol{0}$, using $\int_K \varphi_e\,dx = |K|/3$ from Problem 1(a). Show that $N_M = \mathbb{R} \oplus \operatorname{span}\{\lambda_{a_1} + \lambda_{a_2}\}$, where $a_1$, $a_2$ are the vertices opposite $e$, and conclude $\dim N_M = 2$.

(b) Write out the three-element argument and the propagation step in detail, and prove Lemma 41 for an arbitrary edge-connected macroelement of at least three triangles: a triangle with two interior edges satisfies $\nabla q = \boldsymbol{0}$ (two non-parallel tangential directions), and the pair relation of (a) propagates this property across interior edges.

(c) Verify condition (M2) of Theorem 40 for an arbitrary macroelement partition, using the operator of Theorem 35: matching edge means implies $\int_{\partial M} (\boldsymbol{v} - \Pi_h\boldsymbol{v}) \cdot \boldsymbol{n}\,ds = 0$ for every union of elements $M$, which is condition (F1) for the pressure space $P_0^{\mathrm{disc}}(\mathcal{M}_h)$, with the same Fortin constant.

(d)* Three dimensions: for a vertex $z$ interior to $M$, derive the ring relation $\sum_{K\supset e} |K|\, \nabla q|_K = \mathbf{0}$ for each edge $e$ through $z$, using $\int_K \lambda_i\lambda_j\, dx = |K|/20$ on tetrahedra, and conclude that every tetrahedron with a vertex interior to $M$ satisfies $\nabla q|_K = \mathbf{0}$.

(e)* Prove that every connected triangulation with at least three triangles admits a partition into edge-connected macroelements of three to seven triangles. (Take a spanning tree of the element-adjacency graph, whose maximum degree is three, and repeatedly cut minimal subtrees of size at least three.)

**Problem 3 [T/N] — Solver theory**

The matrices refer to the Hood–Taylor discretization with the preconditioner of slide 27, $\mathsf{P} = \mathrm{diag}(\mathsf{A}, \nu^{-1}\mathsf{M}_p)$; $\mathsf{S} = \mathsf{B}\mathsf{A}^{-1}\mathsf{B}^T$ denotes the pressure Schur complement, and all pressure statements are on the zero-mean subspace.

(a) Prove Theorem 45: for every zero-mean $\mathbf{q} \neq \mathbf{0}$,

$$\beta_h^2 \;\le\; \frac{\nu\, \mathbf{q}^T\mathsf{S}\mathbf{q}}{\mathbf{q}^T\mathsf{M}_p\mathbf{q}} \;\le\; 1.$$

For the upper bound use Lemma 43 ($\|\mathrm{div}\, \boldsymbol{v}\|_0 \le \|\nabla \boldsymbol{v}\|_0$ on $(H_0^1)^2$); for the lower bound use (InfSup$_h$) together with the characterization $\mathbf{q}^T\mathsf{S}\mathbf{q} = \sup_{\mathbf{v}\neq\mathbf{0}} (\mathbf{q}^T\mathsf{B}\mathbf{v})^2/\mathbf{v}^T\mathsf{A}\mathbf{v}$.

(b) Deduce $\|b\| \le 1$ in the norms of the course conventions, and compare with the elementary bound $\|\mathrm{div}\, \boldsymbol{v}\|_0 \le \sqrt{d}\, \|\nabla \boldsymbol{v}\|_0$ obtained from $\mathrm{div}\, \boldsymbol{v} = \mathrm{tr}(\nabla \boldsymbol{v})$ and the Cauchy–Schwarz inequality. Which step in the proof of Lemma 43 removes the factor $\sqrt{d}$?

(c) [N] On the $8 \times 8$ triangulation, assemble $\mathsf{S}$ with the exact $\mathsf{A}^{-1}$ (adapt the spectrum cell of `L4B`), compute the eigenvalues of $\nu\, \mathsf{M}_p^{-1}\mathsf{S}$ on the zero-mean subspace, and verify the interval of Theorem 45 with the value of $\beta_h$ from Demonstration 1b.

(d)* Derive the eigenvalue inclusion of Corollary 46 from the scalar relation $\lambda(\lambda - 1) = \sigma$, where $\sigma$ ranges over the spectrum computed in (c). Then replace the pressure block $\nu^{-1}\mathsf{M}_p$ by the exact Schur complement $\mathsf{S}$ and verify, numerically and from the same relation with $\sigma \equiv 1$, that the preconditioned matrix has exactly the three eigenvalues $\{1, (1 \pm \sqrt{5})/2\}$ (Murphy–Golub–Wathen).

**Problem 4 [N] — Measurements**

All parts run on the notebooks as distributed; the self-study cells named below are complete and runnable.

(a) Run the Demonstration 3 cells of `L4B` (MINRES under refinement on the lid-driven cavity, exact block solves). Compare the counts with the reference-implementation record quoted on slide 29, which is measured on the manufactured problem of Demonstration 1a on triangular meshes. Explain the difference and the shared assertion: the two linear systems differ, and the flatness of the count under refinement is common to both.

(b) Run the $\nu$-sweep cell of `L4B` ($\nu \in \{1, 10^{-2}, 10^{-4}\}$ at fixed $N$). State what Theorem 45 implies for the preconditioned spectrum across $\nu$, and explain which ingredient of the experiment can nevertheless make the counts vary; backup frame B5 records the manufactured-problem case.

(c) Run Demonstration 1c(i) and the $P_1$–$P_1$ kernel count of Demonstration 1b. Report $\max |p_h|$ and $\dim\ker \mathsf{B}^T$ at $N = 8$, and explain, with the range argument of slide 23, why the $\varepsilon$-shifted solve produces a bounded pressure for the manufactured load.

(d)* Measure $\beta_h$ for Hood–Taylor at $N = 8, 16, 24$ (Demonstration 1b) and report the trend. This part has no exact reference value: unlike the one-dimensional constant $\pi/\sqrt{1+\pi^2}$ of Lecture 2 and the eigenvalue formula of Lecture 3, Exercise 4(d)*, the exact continuous inf-sup constant of the square is not provided in this course; the asymmetry is intentional.

(e)* Reproduce the gradient-forcing table of slide 26 for one pair (self-study cell (c) of `L4A`). Explain why the discrete pressure is independent of $\nu$ and why the discrete velocity scales exactly as $\nu^{-1}$: substitute $\hat{\mathbf{u}} = \nu \mathbf{u}$ in the discrete system with the fixed load $\boldsymbol{f} = \nabla\phi$.

(f)* For the regularized $Q_1$–$P_0$ cavity of backup frame B6 ($\varepsilon$-shift with parameter $\varepsilon$, lifted lid datum $\boldsymbol{g}_h$), test the discrete pressure equation with the checkerboard $\chi \in \ker \mathsf{B}^T$ and derive

$$\langle p_h, \chi \rangle \;=\; -\,\frac{(\operatorname{div} \boldsymbol{g}_h,\ \chi)}{\varepsilon}.$$

Show that only the top row of cells contributes, with the cell value $c_i = \frac{h}{2}(\mathrm{lid}_{i+1} - \mathrm{lid}_i)$, so that the datum component is an alternating sum of lid-node differences. Evaluate it at $N = 32$ for (i) the quartic lid $u_x = 16x^2(1-x)^2$ (a five-line computation) and (ii) the constant lid whose two corner values are set to zero (the sum equals $h$ exactly); divide by $\varepsilon = 10^{-8}$, and compare with the checkerboard amplitudes recorded in Lecture 1 and on backup frame B6.