The Babuška–Brezzi theory

Mixed finite element methods — a crash course · Lecture 2 of 4

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All numerical results in these slides were produced by the verified reference implementation that accompanies the course materials.

The contract of Lecture 1

Question 1

Under which conditions on a and b is the continuous saddle-point problem well posed?

Question 2

Under which conditions does a Galerkin discretization inherit well-posedness, with constants independent of h?

Both questions are answered today. The verification of the inf-sup hypothesis for Stokes is deferred to Lecture 4.

Setting, notation, and one remark on signs

Real Hilbert spaces V, Q; bounded bilinear forms a on $V \times V$ and b on $V \times Q$; $F \in V'$, $G \in Q'$.

$$\text{Find } (u,p) \in V \times Q : \quad \begin{aligned} a(u,v) + b(v,p) &= F(v) \quad && \forall v \in V, \\ b(u,q) &= G(q) \quad && \forall q \in Q. \end{aligned}$$

Operators $B : V \to Q'$ and $B^t : Q \to V'$, $\langle B^t q, v\rangle = b(v,q)$; kernels $K = \ker B$, and $K_h = \{v_h \in V_h : b(v_h, q_h) = 0 \ \forall q_h \in Q_h\}$. After a choice of bases,

$$\begin{pmatrix} \mathsf{A} & \mathsf{B}^T \\ \mathsf{B} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \mathbf{g} \end{pmatrix}.$$

The sign of b is a per-problem convention: $b = +(\operatorname{div}\cdot, \cdot)$ for the mixed Laplacian and $b = -(\operatorname{div}\cdot, \cdot)$ for Stokes. The theory of this lecture is invariant under $b \mapsto -b$. This remark is not repeated.

The finite-dimensional characterization

Proposition 1

Let $\mathsf{A} \in \mathbb{R}^{N\times N}$, $\mathsf{B} \in \mathbb{R}^{M\times N}$, and let the columns of Z form a basis of $\ker \mathsf{B}$. The block matrix $\begin{pmatrix} \mathsf{A} & \mathsf{B}^T \\ \mathsf{B} & 0 \end{pmatrix}$ is invertible if and only if

$$\text{(a)} \quad \mathsf{Z}^T\mathsf{A}\mathsf{Z} \text{ is invertible} \qquad \text{and} \qquad \text{(b)} \quad \ker \mathsf{B}^T = \{\mathbf{0}\}.$$

▶ **board:** Proof on the board, in three named steps. The same three steps carry the entire lecture.

1. **Lifting.** A particular solution of $\mathsf{B}\mathbf{u} = \mathbf{g}$ is $\mathbf{u}_g = \mathsf{B}^T(\mathsf{B}\mathsf{B}^T)^{-1}\mathbf{g}$; the general solution is $\mathbf{u} = \mathbf{u}_g + \mathsf{Z}\mathbf{w}$.
2. **Kernel problem.** The first row tested on $\ker \mathsf{B}$ gives $\mathsf{Z}^T\mathsf{A}\mathsf{Z}\,\mathbf{w} = \mathsf{Z}^T(\mathbf{f} - \mathsf{A}\mathbf{u}_g)$.
3. **Multiplier.** $\mathsf{B}^T\mathbf{p} = \mathbf{f} - \mathsf{A}\mathbf{u}$ is solvable because the right-hand side is orthogonal to $\ker \mathsf{B}$ by Step 2, and $\operatorname{ran} \mathsf{B}^T = (\ker \mathsf{B})^\perp$; uniqueness of $\mathbf{p}$ is (b).

Consequences, and the program of the lecture

- Lemma 1 of Lecture 1 is the necessity of condition (b): a nonzero $\ker \mathsf{B}^T$ (a spurious mode) makes the system singular.
- The two failure channels of Lecture 1 in matrix language:

pair (1D)	(a): $\mathsf{Z}^T\mathsf{A}\mathsf{Z}$	(b): $\ker \mathsf{B}^T$
P_1–P_1	invertible	$\neq \{\mathbf{0}\}$ (singular system)
P_2–P_0	invertible at each n, degenerates	$= \{\mathbf{0}\}$
P_1–P_0	uniformly invertible	$= \{\mathbf{0}\}$

- Invertibility at each fixed h is not the right question (Lecture 1, closing). The correct question is quantitative.

The remainder of the lecture extends Proposition 1 to Hilbert spaces and makes all constants explicit and uniform in h: condition (a) becomes the kernel hypothesis (ElKer)**, condition (b) becomes the inf-sup hypothesis** (InfSup)**.**

The inf-sup dictionary

Lemma 2 (inf-sup dictionary)

Let $\beta > 0$. The following statements are equivalent.

(i) $\inf_{0 \neq q \in Q} \sup_{0 \neq v \in V} \dfrac{b(v,q)}{\|v\|_V \, \|q\|_Q} \;\geq\; \beta.$

(ii) $\|B^t q\|_{V'} \;\geq\; \beta \, \|q\|_Q$ for all $q \in Q$.

(iii) For every $G \in Q'$ there exists $v_G \in V$ with $Bv_G = G$ and $\|v_G\|_V \leq \|G\|_{Q'}/\beta$.

(iv) In (iii), v_G can be chosen in $K^\perp$; it is then unique, of minimal norm, and depends linearly on G.

Reading: (ii) is the quantitative injectivity of B^t; (iii) is the quantitative surjectivity of B. In Hilbert spaces this equivalence replaces the closed-range theorem; the proof below uses only the Riesz representation and the Lax–Milgram lemma.

Proof of Lemma 2, part I

(i) $\Leftrightarrow$ (ii). By definition of the dual norm, $\sup_v b(v,q)/\|v\|_V = \|B^t q\|_{V'}$; (i) is (ii) after dividing by $\|q\|_Q$.

(ii) $\Rightarrow$ (iii). Let $Rq \in V$ be the Riesz representative of $B^t q$, so that

$$(Rq, v)_V = b(v,q) \quad \forall v \in V, \qquad \|Rq\|_V = \|B^t q\|_{V'} \ \geq \ \beta \|q\|_Q.$$

The bilinear form $s(q,r) := b(Rq, r)$ on $Q \times Q$ is bounded, $|s(q,r)| \leq \|b\| \, \|Rq\|_V \, \|r\|_Q \leq \|b\|^2 \|q\|_Q \|r\|_Q$, and coercive,

$$s(q,q) = b(Rq,q) = (Rq,Rq)_V \ \geq \ \beta^2 \|q\|_Q^2.$$

By Lax–Milgram, for each $G \in Q'$ there is a unique q_G with $s(q_G, r) = G(r)$ for all r. Set $v_G := Rq_G$. Then $Bv_G = G$, and

$$\|v_G\|_V^2 = s(q_G, q_G) = G(q_G) \leq \|G\|_{Q'} \|q_G\|_Q \leq \|G\|_{Q'}^2 / \beta^2,$$

using $\beta^2 \|q_G\|_Q^2 \leq s(q_G, q_G) \leq \|G\|_{Q'} \|q_G\|_Q$. $\square$ (part I)

Proof of Lemma 2, part II

(iv). $(Rq, v)_V = b(v, q) = 0$ for $v \in K$, so $\operatorname{ran} R \subset K^\perp$ and $v_G = Rq_G \in K^\perp$. Two liftings of the same G differ by an element of K; hence the $K^\perp$ lifting is unique and of minimal norm, and $G \mapsto v_G$ is linear because $G \mapsto q_G$ is.

R **is a bijection from** Q **onto** $K^\perp$ (the tool for the multiplier step in the well-posedness proof below). Indeed, R is injective with closed range by (ii) and completeness; if $v \in K^\perp$ is orthogonal to $\operatorname{ran} R$, then $b(v, q) = (Rq, v)_V = 0$ for all q, so $v \in K$ and therefore $v = 0$.

(iii) $\Rightarrow$ (i). Given $q \neq 0$, take $G := (\cdot, q)_Q$, so $\|G\|_{Q'} = \|q\|_Q$. Then

$$\sup_v \frac{b(v,q)}{\|v\|_V} \;\geq\; \frac{b(v_G, q)}{\|v_G\|_V} = \frac{\langle G, q\rangle}{\|v_G\|_V} = \frac{\|q\|_Q^2}{\|v_G\|_V} \;\geq\; \beta\, \|q\|_Q. \qquad \square$$

The dictionary in matrix form

Fix bases $\{\varphi_i\}$ of V_h and $\{\psi_j\}$ of Q_h, and set

$$(\mathsf{G}_V)_{ik} = (\varphi_k, \varphi_i)_V, \qquad (\mathsf{G}_Q)_{jl} = (\psi_l, \psi_j)_Q,$$
$$(\mathsf{A})_{ik} = a(\varphi_k, \varphi_i), \qquad (\mathsf{B})_{ji} = b(\varphi_i, \psi_j),$$

so that $\|v_h\|_V^2 = \mathbf{v}^T \mathsf{G}_V \mathbf{v}$ and $\|q_h\|_Q^2 = \mathbf{q}^T \mathsf{G}_Q \mathbf{q}$ for the coefficient vectors. Then

$$\beta_h = \sigma_{\min}\left(\mathsf{G}_V^{-1/2}\, \mathsf{B}^T\, \mathsf{G}_Q^{-1/2}\right);$$

equivalently, β_h^2 is the smallest eigenvalue of

$$\mathsf{B}\, \mathsf{G}_V^{-1} \mathsf{B}^T\, \mathbf{q} = \lambda\, \mathsf{G}_Q\, \mathbf{q}.$$

With the columns of Z spanning $\ker \mathsf{B}$, moreover, $\alpha_h = \lambda_{\min}$ of $\mathsf{Z}^T \mathsf{A} \mathsf{Z}\, \mathbf{w} = \mu\, \mathsf{Z}^T \mathsf{G}_V \mathsf{Z}\, \mathbf{w}$.
Proof: Exercise 1(a). Zero singular values correspond exactly to the spurious modes $\ker \mathsf{B}^T$. ▶ **live demonstration:** Cells 4 and 7 of `L2_constants.ipynb` compute exactly these two quantities; every number shown later today is one of them.

The Brezzi theorem

Theorem 3 (Brezzi, 1974)

Assume that a and b are bounded and that

$$a(v,v) \;\geq\; \alpha\,\|v\|_V^2 \quad \text{for all } v \in K, \tag{ElKer}$$

$$\inf_{0\neq q\in Q}\;\sup_{0\neq v\in V}\;\frac{b(v,q)}{\|v\|_V\,\|q\|_Q} \;\geq\; \beta \;>\; 0. \tag{InfSup}$$

Then the saddle-point problem has a unique solution (u,p) for every $(F,G) \in V' \times Q'$, and

$$\begin{aligned}
\|u\|_V \;&\leq\; \frac{1}{\alpha}\|F\|_{V'} + \frac{1}{\beta}\Big(1+\frac{\|a\|}{\alpha}\Big)\|G\|_{Q'},\\
\|p\|_Q \;&\leq\; \frac{1}{\beta}\Big(1+\frac{\|a\|}{\alpha}\Big)\|F\|_{V'} + \frac{\|a\|}{\beta^2}\Big(1+\frac{\|a\|}{\alpha}\Big)\|G\|_{Q'}.
\end{aligned}$$

The constants are explicit but not sharp. The terminology follows Boffi–Brezzi–Fortin: ellipticity in the kernel, inf-sup condition.

Proof of Theorem 3: lifting and kernel problem

Step 1 (lifting). By (InfSup) and Lemma 2(iii)–(iv) there is $u_G \in K^\perp$ with $Bu_G = G$ and $\|u_G\|_V \le \|G\|_{Q'}/\beta$.

Step 2 (kernel problem). Seek $u_0 \in K$ with

$$a(u_0, v) = F(v) - a(u_G, v) \qquad \forall v \in K.$$

The right-hand side is a bounded functional on K of norm at most $\|F\|_{V'} + \|a\|\|u_G\|_V$; by (ElKer) and Lax–Milgram on the closed subspace K, u_0 exists, is unique, and

$$\|u_0\|_V \ \le\ \frac{1}{\alpha}\big(\|F\|_{V'} + \|a\|\,\|u_G\|_V\big).$$

Set $u := u_0 + u_G$; then $Bu = G$, and

$$\|u\|_V \ \le\ \frac{1}{\alpha}\|F\|_{V'} + \frac{1}{\beta}\Big(1 + \frac{\|a\|}{\alpha}\Big)\|G\|_{Q'}.$$

Proof of Theorem 3: multiplier and uniqueness

Step 3 (multiplier). The residual $r(v) := F(v) - a(u, v)$ satisfies $r|_K = 0$ by Step 2, so its Riesz representative ρ lies in $K^\perp$. Since $R : Q \to K^\perp$ is a bijection (Lemma 2, part II), there is a unique p with $Rp = \rho$, that is,

$$b(v, p) = (Rp, v)_V = r(v) \qquad \forall v \in V,$$

and $\beta \|p\|_Q \leq \|Rp\|_V = \|r\|_{V'} \leq \|F\|_{V'} + \|a\| \|u\|_V$. Inserting the bound on $\|u\|_V$ gives the stated estimate for $\|p\|_Q$.

Uniqueness. For $F = 0$, $G = 0$: $u \in K$ and, testing with $v = u$, $a(u, u) = -b(u, p) = 0$, so $u = 0$ by (ElKer); then $B^t p = 0$, so $p = 0$ by Lemma 2(ii). □

One proof, three habitats: the same three steps proved Proposition 1, prove Theorem 3, and are repeated verbatim for the discrete problem.

The mixed Laplacian, hypothesis (ElKer)

$V = H(\mathrm{div};\Omega)$, $Q = L^2(\Omega)$, $a(\boldsymbol{\sigma},\boldsymbol{\tau}) = (\kappa^{-1}\boldsymbol{\sigma},\boldsymbol{\tau})$, $b(\boldsymbol{\tau},v) = (\operatorname{div}\boldsymbol{\tau},v)$; $0 < \kappa_- \le \kappa \le \kappa_+$.

Proposition 4

$K = \{\boldsymbol{\tau} \in H(\mathrm{div};\Omega) : \operatorname{div}\boldsymbol{\tau} = 0\}$, and (ElKer) holds with $\alpha = 1/\kappa_+$.

Proof. $b(\boldsymbol{\tau},v) = 0$ for all $v \in L^2$ means $(\operatorname{div}\boldsymbol{\tau},v) = 0$ for all v, hence $\operatorname{div}\boldsymbol{\tau} = 0$. On K, $\|\boldsymbol{\tau}\|_{\mathrm{div}} = \|\boldsymbol{\tau}\|_0$, and

$$a(\boldsymbol{\tau},\boldsymbol{\tau}) = \int_\Omega \kappa^{-1}|\boldsymbol{\tau}|^2 \;\ge\; \kappa_+^{-1}\|\boldsymbol{\tau}\|_0^2 = \kappa_+^{-1}\|\boldsymbol{\tau}\|_{\mathrm{div}}^2. \qquad \square$$

Remark: the form a is not coercive on V, because it contains no control of the divergence. Coercivity holds exactly on the constrained subspace K; this is the situation for which (ElKer) is formulated.

The mixed Laplacian, hypothesis (InfSup)

Proposition 5

(InfSup) holds on $H(\mathrm{div};\Omega) \times L^2(\Omega)$ with $\beta \geq (1 + C_P^2)^{-1/2}$, where C_P is the Poincaré constant of Ω. The bound does not depend on κ.

Proof. Given $v \in L^2(\Omega)$, let $w \in H_0^1(\Omega)$ solve $-\Delta w = v$ and set $\boldsymbol{\tau} := -\nabla w$. Then $\mathrm{div}\,\boldsymbol{\tau} = v$, so $b(\boldsymbol{\tau}, v) = \|v\|_0^2$, and $\|\boldsymbol{\tau}\|_0 = \|\nabla w\|_0 \leq C_P \|v\|_0$ from $\|\nabla w\|_0^2 = (v, w) \leq C_P \|v\|_0 \|\nabla w\|_0$. Hence $\|\boldsymbol{\tau}\|_{\mathrm{div}}^2 \leq (1 + C_P^2)\|v\|_0^2$ and $b(\boldsymbol{\tau}, v)/\|\boldsymbol{\tau}\|_{\mathrm{div}} \geq (1 + C_P^2)^{-1/2}\|v\|_0$. □

One dimension. For $\Omega = (0,1)$ the lifting is the antiderivative $\tau(x) = \int_0^x v$, which gives $\beta \geq \sqrt{2/3}$ by a two-line computation; the exact value is $\beta = \pi/\sqrt{1+\pi^2} \approx 0.9529$ (Exercise 2(e)).

The two model problems are complementary

	mixed Laplacian	Stokes
a coercive on	only K $(\alpha = 1/\kappa_+)$	all of V $(\alpha = \nu)$
(InfSup)	elementary (Poisson lifting)	deep; Lecture 4
difficult hypothesis	(ElKer)	(InfSup)

For Stokes, $V = (H_0^1(\Omega))^d$ with the norm $\|\nabla \boldsymbol{v}\|_0$, and $a(\boldsymbol{v}, \boldsymbol{v}) = \nu \|\nabla \boldsymbol{v}\|_0^2$, so (ElKer) holds on all of V with $\alpha = \nu$; the entire difficulty is (InfSup) for $b(\boldsymbol{v}, q) = -(\operatorname{div} \boldsymbol{v}, q)$ on $(H_0^1)^d \times L_0^2$.

Each model problem exercises exactly one of the two hypotheses.

The discrete problem

$V_h \subset V$, $Q_h \subset Q$; same equations with (u_h, p_h), tested on (V_h, Q_h). Discrete kernel: $K_h = \{v_h \in V_h : b(v_h, q_h) = 0 \ \forall q_h \in Q_h\}$.

Warning: $K_h \not\subset K$ in general

P_2–P_0 in one dimension: an element bubble b_K satisfies $(b'_K, q_h) = 0$ for all $q_h \in P_0$, because b'_K has zero mean on each element, but $b'_K \neq 0$. Hence $b_K \in K_h \setminus K$: the discrete kernel contains functions that violate the continuous constraint. Consequently, (ElKer) does not transfer to K_h by restriction.

Discrete hypotheses, for a family $(V_h, Q_h)_h$:

$$a(v_h, v_h) \ \geq \ \alpha_h \, \|v_h\|_V^2 \quad \forall v_h \in K_h, \qquad (\mathsf{ElKer}_h)$$

$$\inf_{0 \neq q_h \in Q_h} \ \sup_{0 \neq v_h \in V_h} \frac{b(v_h, q_h)}{\|v_h\|_V \, \|q_h\|_Q} \ \geq \ \beta_h. \qquad (\mathsf{InfSup}_h)$$

The requirement is a uniform bound: $\alpha_h \geq \alpha_* > 0$ and $\beta_h \geq \beta_* > 0$ for the whole family, not positivity at each h.

Discrete stability

Theorem 6

Assume (ElKer_h) and (InfSup_h). Then the discrete problem has a unique solution (u_h, p_h) for every (F, G), and the bounds of Theorem 3 hold with (α, β) replaced by (α_h, β_h) and the dual norms taken on V_h, Q_h.

Proof. Identical to the proof of Theorem 3 with (V, Q) replaced by (V_h, Q_h): the dictionary and the Lax–Milgram argument do not refer to the dimension of the spaces. □

This is why the dictionary was proved in Hilbert-space generality: the finite-dimensional, the continuous, and the discrete statements are one proof in three habitats.

If $\alpha_h \to 0$ or $\beta_h \to 0$, Theorem 6 remains true at each h and degenerates as a family; the constants make the degeneration quantitative. Demo 1 measures it.

Quasi-optimality

Theorem 7 (error estimates)

Assume (ElKer$_h$) and (InfSup$_h$), and let (u,p), (u_h,p_h) solve the continuous and the discrete problem. Then

$$
\begin{aligned}
\|u-u_h\|_V \;&\le\; \Big(1+\frac{\|a\|}{\alpha_h}\Big)\Big(1+\frac{\|b\|}{\beta_h}\Big)\inf_{v_h\in V_h}\|u-v_h\|_V \\
&\quad +\; \frac{\|b\|}{\alpha_h}\inf_{q_h\in Q_h}\|p-q_h\|_Q, \\
\|p-p_h\|_Q \;&\le\; \Big(1+\frac{\|b\|}{\beta_h}\Big)\inf_{q_h\in Q_h}\|p-q_h\|_Q \;+\; \frac{\|a\|}{\beta_h}\,\|u-u_h\|_V.
\end{aligned}
$$

Reading: best-approximation errors multiplied by constants that blow up as $\alpha_h \to 0$ or $\beta_h \to 0$. The β_h-sensitivity of the multiplier is partly hidden in the second term, where the estimate for $\|u-u_h\|_V$ enters: substituting it there shows that the error in p carries $1/\beta_h$ twice (order β_h^{-2}). A deficient inf-sup constant degrades the multiplier more severely than the V-variable.

Proof: constrained approximation and kernel estimate

Step 1. Fix $w_h \in V_h$. By (InfSup$_h$) and the discrete dictionary (Lemma 2(iii) on (V_h, Q_h)) there is $r_h \in V_h$ with

$$b(r_h, q_h) = b(u - w_h, q_h) \ \ \forall q_h \in Q_h, \qquad \|r_h\|_V \le \frac{\|b\|}{\beta_h}\|u - w_h\|_V.$$

Then $v_h^* := w_h + r_h$ satisfies $b(v_h^*, q_h) = b(u, q_h) = G(q_h)$ for all q_h, and $\|u - v_h^*\|_V \le \big(1 + \|b\|/\beta_h\big)\|u - w_h\|_V$.

Step 2. $e_h := u_h - v_h^* \in K_h$. By (ElKer$_h$), the error equations $a(u - u_h, e_h) + b(e_h, p - p_h) = 0$, and $b(e_h, p_h - q_h) = 0$ for every $q_h \in Q_h$, give

$$\alpha_h\|e_h\|_V^2 \ \le \ a(e_h, e_h) = a(u - v_h^*, e_h) + b(e_h, p - q_h)$$

$$\implies \quad \alpha_h\|e_h\|_V \ \le \ \|a\|\,\|u - v_h^*\|_V + \|b\|\,\|p - q_h\|_Q.$$

Proof of Theorem 7: assembly and pressure estimate

Step 3. Triangle inequality and Steps 1–2:

$$\begin{aligned}\|u-u_h\|_V &\le \|u-v_h^*\|_V + \|e_h\|_V \\ &\le \Big(1+\frac{\|a\|}{\alpha_h}\Big)\|u-v_h^*\|_V + \frac{\|b\|}{\alpha_h}\|p-q_h\|_Q.\end{aligned}$$

Inserting the Step 1 bound $\|u-v_h^*\|_V \le (1+\|b\|/\beta_h)\|u-w_h\|_V$ gives a bound valid for every pair (w_h, q_h), with a left-hand side independent of both; the infimum over both, split over the two terms, gives the first estimate.

Step 4. For any $q_h \in Q_h$, by (InfSup$_h$) and the two first equations ($b(v_h,p_h) = F(v_h) - a(u_h,v_h)$ and $b(v_h,p) = F(v_h) - a(u,v_h)$ for $v_h \in V_h \subset V$),

$$\begin{aligned}\beta_h\|p_h-q_h\|_Q &\le \sup_{v_h}\frac{b(v_h,p_h-q_h)}{\|v_h\|_V} \\ &\le \|b\|\,\|p-q_h\|_Q + \|a\|\,\|u-u_h\|_V,\end{aligned}$$

and the triangle inequality gives the second estimate. □

Decoupling the multiplier

Corollary 8

If $K_h \subset K$, the multiplier term disappears from the first estimate:

$$\|u - u_h\|_V \;\leq\; \left(1 + \frac{\|a\|}{\alpha_h}\right)\left(1 + \frac{\|b\|}{\beta_h}\right) \inf_{v_h \in V_h} \|u - v_h\|_V .$$

Proof. In Step 2, $e_h \in K_h \subset K$ gives $b(e_h, p - q_h) = \langle Be_h, p - q_h \rangle = 0$.
□

Remark (the balance between the spaces)

Enlarging Q_h with V_h fixed shrinks K_h: α_h does not decrease, and β_h does not increase. Enlarging V_h with Q_h fixed has the two opposite effects. The hypotheses (ElKer$_h$) and (InfSup$_h$) therefore impose opposite requirements on the relative size of the two spaces; a stable pair is a balance, not a maximization. Precise statements and verification against the measured table: Exercise 1(c). This is the quantitative form of the balance slide of Lecture 1.

Guaranteeing $K_h \subset K$

Observation 9

Assume $b(v, q) = (\Lambda v, q)_Q$ for a bounded linear operator $\Lambda : V \to Q$. If $\Lambda V_h \subset Q_h$, then $K_h \subset K$.

Proof. For $v_h \in K_h$, the choice $q_h := \Lambda v_h \in Q_h$ gives $\|\Lambda v_h\|_Q^2 = 0$; hence $b(v_h, q) = (\Lambda v_h, q)_Q = 0$ for all $q \in Q$. □

- Mixed Laplacian ($\Lambda = \operatorname{div}$, $Q = L^2$): in one dimension, P_1–P_0 satisfies $\operatorname{div} V_h \subset Q_h$ (derivatives of P_1 are P_0); P_2–P_0 does not (derivatives of P_2 are $P_1 \not\subset P_0$).
- Stokes ($\Lambda = -\operatorname{div}$, $Q = L_0^2$): the same inclusion characterizes pairs whose discrete velocities are exactly divergence free; such pairs exist but are not among the classical ones (Lecture 4, outlook).
- The multiplier space can be very small without damaging the V-approximation; Lecture 3 constructs spaces for which the inclusion holds by design.

Demo 1a: the Lecture 1 suite, measured

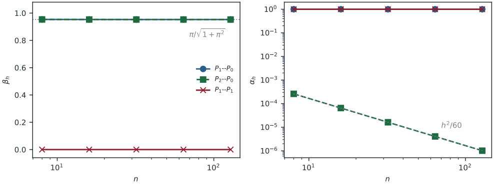
▶ **live demonstration:** `L2_constants.ipynb`, cell 4. Norms: $\|\cdot\|_{\mathrm{div}}$ on V_h, $\|\cdot\|_0$ on Q_h; $n = 8, \ldots, 128$.

pair	β_h	α_h	$\dim K_h$
P_1–P_1	0 (exact)	1 (exact)	1
P_2–P_0	≈ 0.9529, flat	$h^2/60$ (ratio $\uparrow 1/60$)	$n+1$
P_1–P_0	≈ 0.9529, flat	1 (exact)	1

- The two failures are exactly orthogonal: P_1–P_1 has a perfect kernel constant and a zero inf-sup constant; P_2–P_0 has a uniform inf-sup constant and a vanishing kernel constant. The hypotheses (ElKer$_h$) and (InfSup$_h$) are independently necessary.
- Every entry marked exact is a theorem (Exercises 1–2); the bound $\beta_h \geq \sqrt{2/3}$ is the antiderivative lifting; the ratio $1/60$ is the sharp constant measured in Lecture 1.

Demo 1b: constants under refinement

- **live demonstration:** `L2_constants.ipynb`, cell 5.



- The measured β_h of both stable pairs converge from above to the exact continuous value $\pi/\sqrt{1+\pi^2} = 0.952891$.
- Blow-up factor $(1 + \|a\|/\alpha_h)(1 + \|b\|/\beta_h)$ of Theorem 7: for P_2–P_0 it grows from $7.9 \cdot 10^3$ $(n = 8)$ to $2.0 \cdot 10^6$ $(n = 128)$; for P_1–P_0 it is 4.099, flat.
- Only the V-term of Theorem 7 is balanced: $1/\alpha_h \sim h^{-2}$ against $\inf\|\sigma - \tau_h\|_{\mathrm{div}} \sim h^2$ gives $O(1)$; the Q-term pairs the same factor with $\inf\|u - q_h\|_0 \sim h$, so the bound is $O(h^{-1})$ — it permits the flatlines 0.9129 and 0.8333 without predicting them (backup B1).

Necessity of the two hypotheses

Theorem 10 (necessity, symmetric case)

Let a be symmetric positive semi-definite. Assume that for every $(F, G) \in V_h' \times Q_h'$ the discrete problem has a solution with $\|u_h\|_V + \|p_h\|_Q \leq C\,(\|F\|_{V_h'} + \|G\|_{Q_h'})$. Then $\beta_h \geq 1/C$ and $\alpha_h \geq 1/(C^2\|a\|)$.

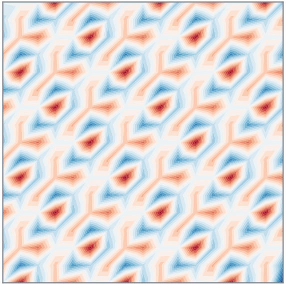
Proof. Solvability for all data in finite dimensions implies uniqueness. (1) Given q_h, take $F = 0$, $G = (q_h, \cdot)_Q$; the second equation with $r_h = q_h$ gives $b(u_h, q_h) = \|q_h\|_Q^2$, so $\sup_v b(v, q_h)/\|v\|_V \geq \|q_h\|_Q^2/\|u_h\|_V \geq \|q_h\|_Q/C$. (2) Given $w_h \in K_h$, take $F = a(\cdot, w_h)$, $G = 0$; then $(w_h, 0)$ is the solution, and by the Cauchy–Schwarz inequality for the semi-inner product a, $\|w_h\|_V \leq C\|F\|_{V_h'} \leq C\,\|a\|^{1/2} a(w_h, w_h)^{1/2}$, hence $a(w_h, w_h) \geq \|w_h\|_V^2/(C^2\|a\|)$. $\square$

Both model problems have symmetric positive semi-definite a: stability and the two hypotheses are the same property.

Demo 2a: the triangular P_1–P_1 pair

▶ **live demonstration:** `L2_constants.ipynb`, cell 7. Stokes setting; V-norm $\|\nabla\cdot\|_0$; structured mesh, either diagonal.

- Weighted eigenvalue problem of the dictionary at $N = 16$: the number of numerically zero eigenvalues is **8** — the constant and seven further exact spurious pressure modes.
- The count is 8 for every $N \geq 8$, for both diagonals, and for both parities of N.
- This count is easy to predict incorrectly; the verification script is part of the distributed bundle.



Demo 2b: the essential constant; both species at once

▶ **live demonstration:** `L2_constants.ipynb`, cell 8; measured decay rate 0.97 ($N = 24$–40). Let $\widetilde{Q}_h$ be the orthogonal complement of the exact modes in Q_h and $\widetilde{\beta}_h$ the inf-sup constant of $(V_h, \widetilde{Q}_h)$.

N	8	12	16	24	32	40
$\widetilde{\beta}_h$	0.0717	0.0522	0.0405	0.0276	0.0209	0.0168

- The pair fails through both mechanisms of Lecture 1: exact spurious modes ($\beta_h = 0$, species (i)), and uniform decay of the essential constant — $(V_h, \widetilde{Q}_h)$ has no exact mode and $\widetilde{\beta}_h \to 0$ (species (ii)). Either alone contradicts uniform stability by Theorem 10.
- The mode count is fragile: 8 (structured), 5 (randomly perturbed vertices), 7 (alternating diagonals, perturbed). The decay is robust.
- The one-dimensional P_1–P_1 pair has the same two-layer structure: one exact mode, and a second constant decaying like $O(h)$.

Fortin's criterion

Theorem 11 (Fortin)

Assume (InfSup) with constant β, and suppose there are linear operators $\Pi_h : V \to V_h$ such that, uniformly in h,

$$b(\Pi_h v, q_h) = b(v, q_h) \quad \forall v \in V,\ q_h \in Q_h, \tag{F1}$$

$$\|\Pi_h v\|_V \ \le\ C_\Pi \|v\|_V \quad \forall v \in V. \tag{F2}$$

Then (InfSup$_h$) holds with $\beta_h \geq \beta / C_\Pi$.

▶ **board:** Proof on the board; four lines. For $q_h \neq 0$ and v with $b(v, q_h) > 0$, (F1) gives $\Pi_h v \neq 0$ and

$$\sup_{v_h} \frac{b(v_h, q_h)}{\|v_h\|_V} \ \geq\ \frac{b(\Pi_h v, q_h)}{\|\Pi_h v\|_V} = \frac{b(v, q_h)}{\|\Pi_h v\|_V} \ \geq\ \frac{1}{C_\Pi} \frac{b(v, q_h)}{\|v\|_V};$$

take the supremum over v and apply (InfSup). A discrete inf-sup proof reduces to the construction of one operator. Exercise 2 constructs it for the working pair of Lecture 1.

First strengthening: a restricted domain

Corollary 12 (Fortin with a restricted domain)

Let $W \subseteq V$ be a subspace with norm $\|\cdot\|_W$, and assume

$$\sup_{0 \neq w \in W} \frac{b(w,q)}{\|w\|_W} \;\geq\; \beta_W \, \|q\|_Q \quad \forall q \in Q. \qquad (\mathsf{InfSup}_W)$$

If linear operators $\Pi_h : W \to V_h$ satisfy ($\mathsf{F1}_W$) on W and $\|\Pi_h w\|_V \leq C_\Pi \|w\|_W$ ($\mathsf{F2}_W$), then $\beta_h \geq \beta_W / C_\Pi$.

Proof. The same four lines, with the supremum over $w \in W$. $\square$

Use in Lecture 3: on a convex polytope, the lifting of Proposition 5 satisfies the stronger bound $\|\boldsymbol{\tau}\|_1 \leq C_{\mathrm{reg}} \|v\|_0$ (elliptic regularity, stated without proof), so (InfSup_W) holds with $W = (H^1(\Omega))^d$. The interpolation operator of the spaces constructed in Lecture 3 is not bounded on all of $H(\mathrm{div};\Omega)$; it satisfies ($\mathsf{F1}_W$)–($\mathsf{F2}_W$) on this W. Corollary 12 is the form in which the criterion is applied there.

Second strengthening: construction by composition

Lemma 13 (composition)

Let $\Pi_h^1, \Pi_h^2 : V \to V_h$ be linear with, uniformly in h:
(i) $\|\Pi_h^1 v\|_V \le C_1 \|v\|_V$;
(ii) $b(\Pi_h^2 v, q_h) = b(v, q_h)$ for all v, q_h;
(iii) $\|\Pi_h^2 (v - \Pi_h^1 v)\|_V \le C_2 \|v\|_V$.
Then $\Pi_h := \Pi_h^1 + \Pi_h^2 (I - \Pi_h^1)$ satisfies (F1)–(F2) with $C_\Pi = C_1 + C_2$.

Proof. (F1): $b(\Pi_h v, q_h) = b(\Pi_h^1 v, q_h) + b(v - \Pi_h^1 v, q_h) = b(v, q_h)$ by (ii). (F2): triangle inequality. ☐

Division of labour: Π_h^1 provides boundedness without moment matching; Π_h^2 repairs the constraint moments and is required to be bounded only on the residual $v - \Pi_h^1 v$.

Use in Lecture 4: Π_h^2 an elementwise bubble correction; this proves stability of the MINI element. Bubbles in the P_2–P_0 kernel destroyed coercivity; in Lecture 4 they supply the missing stability.

The converse, and the two announced uses

Theorem 14 (converse of Fortin's criterion)

If $\beta_h \geq \beta_* > 0$ for all h, then Fortin operators exist with $C_\Pi \leq \|b\|/\beta_*$.

Proof: Exercise 3. Idea: apply Lemma 2(iii)–(iv) to (V_h, Q_h) with the data $G_v := b(v,\cdot)|_{Q_h}$; the minimal-norm lifting is unique and therefore linear in v.

Uniform inf-sup stability is equivalent to the existence of a uniformly bounded Fortin family. To prove stability, construct an operator; to prove instability, exhibit a spurious mode or a degenerating sequence $\beta_h \to 0$.

Lecture 3	Corollary 12, $W = (H^1)^d$	canonical interpolation of the new spaces; (F1$_W$) by commutation
Lecture 4	Lemma 13	Clément-type Π_h^1 + bubble Π_h^2; stability of MINI

The toolkit

Question 1	Theorem 3	(ElKer) + (InfSup) $\Rightarrow$ well-posed, explicit constants
	Propositions 4–5	mixed Laplacian: both hypotheses verified
	Lecture 4	Stokes: (InfSup) stated there
Question 2	Theorem 6	(ElKer$_h$) + (InfSup$_h$) $\Rightarrow$ discrete stability
	Theorem 7	quasi-optimal errors; $1/\alpha_h$, $1/\beta_h$ visible
	Corollary 8, Obs. 9	$K_h \subset K$: V-error decoupled from Q_h
	Theorem 10	the hypotheses are also necessary
	Theorems 11–14	Fortin: the practical test, and its converse

Both questions of the Lecture 1 contract are answered. The remainder of the course constructs pairs that satisfy the hypotheses (Lectures 3 and 4) and discusses solvers for the resulting linear systems (Lecture 4).

The Lecture 1 evidence, completed

pair	Lecture 1 observation	Lecture 2 statement
P_1–P_1 (1D)	singular $\forall n$; mode χ_h	$\beta_h = 0$, $\alpha_h = 1$ (exact)
P_2–P_0 (1D)	errors $\to 0.9129/0.8333$	$\beta_h \approx 0.953$; $\alpha_h = h^2/60$
P_1–P_0 (1D)	clean rates	$\alpha_h = 1$; $\beta_h \to \pi/\sqrt{1+\pi^2}$; $K_h \subset K$
Q_1–P_0 (2D)	two kernel modes	$\beta_h = 0$; kernel: constant + checkerboard
P_1–P_1 (tri.)	Exercise 4(c)	$\beta_h = 0$: 8 modes (5 generic); $\widetilde{\beta}_h \sim O(h)$

Every number Lecture 1 measured now has a theorem attached. For the working pair, the exercise sheet assembles the complete proof of everything that was observed.

Exercises; Lecture 3

Exercise sheet (theory [T] and notebook [N] problems; starred parts optional):

1. [T] Matrix form of the constants; monotonicity in V_h and Q_h.
2. [T] A Fortin operator for the working pair of Lecture 1; the exact one-dimensional constant $\pi/\sqrt{1+\pi^2}$.
3. [T] The converse of Fortin's criterion; a nonexistence certificate for P_1–P_1.
4. [N] Measuring the theory: the constants table; the triangular pair; worst-case data; the exact modes (open-ended part).

Lecture 3 constructs, for $d \geq 2$, spaces that satisfy $\operatorname{div} V_h \subset Q_h$ by design, together with an interpolation operator with the commuting property required by (InfSup$_W$) — and identifies the pair of Lecture 1.

Reading: Boffi–Brezzi–Fortin, Chapters 4–5 (primary); Gatica, Chapter 2 (gentle companion); Brezzi 1974 (historical).

Backup slides

Backup: the estimates are upper bounds

For the P_1–P_0 pair, Theorem 7 with Corollary 8 and standard interpolation gives $\|\sigma - \sigma_h\|_{\mathrm{div}} = O(h)$. Lecture 1 measured, in addition: $\|\sigma - \sigma_h\|_0 = O(h^2)$ and nodal exactness.

quantity	proved today	measured in Lecture 1
$\|\sigma - \sigma_h\|_{\mathrm{div}}$	$O(h)$	rate 1.00
$\|\sigma - \sigma_h\|_0$	$O(h)$ (as part of the above)	rate 2.00
nodal values	—	exact

Quasi-optimality bounds the error from above; superconvergence beyond the bound is compatible with the estimate and requires separate, one-dimensional arguments (Lecture 1, frame 12). The honest multi-dimensional rate is the one proved.

Backup: the single-form point of view

Define $X := V \times Q$ and

$$\mathcal{A}\big((u,p),(v,q)\big) := a(u,v) + b(v,p) + b(u,q),$$
$$\mathcal{F}(v,q) := F(v) + G(q).$$

The saddle-point problem reads: find $x \in X$ with $\mathcal{A}(x,y) = \mathcal{F}(y)$ for all $y \in X$. The Banach–Nečas–Babuška theorem characterizes well-posedness by an inf-sup condition on $\mathcal{A}$; under the symmetry of the block structure, that condition is equivalent to (ElKer) + (InfSup), with constants that are polynomial expressions in α, β, $\|a\|$, $\|b\|$.

The two-constant formulation is preferred in this course because the two constants degenerate independently (Demo 1) and are measured separately. Reference: Ern–Guermond, Finite Elements II, Part on saddle-point problems.

Backup: proof sketch for the converse (Theorem 14)

Assume $\beta_h \geq \beta_*$. For $v \in V$ define $G_v \in Q_h'$ by $G_v(q_h) := b(v, q_h)$; then $\|G_v\|_{Q_h'} \leq \|b\|\|v\|_V$.

By Lemma 2(iii)–(iv) applied to the pair (V_h, Q_h), there is a unique $\Pi_h v \in K_h^\perp \cap V_h$ with

$$b(\Pi_h v, q_h) = b(v, q_h) \quad \forall q_h \in Q_h,$$
$$\|\Pi_h v\|_V \ \leq\ \|G_v\|_{Q_h'}/\beta_h \ \leq\ (\|b\|/\beta_*)\,\|v\|_V.$$

Linearity of $v \mapsto \Pi_h v$ follows from the uniqueness of the minimal-norm lifting and the linearity of $v \mapsto G_v$. □

Full details are Exercise 3; part (e) uses the spurious mode χ_h as a two-line nonexistence certificate for P_1–P_1.

Backup: worst-case data for P_2–P_0

With $w_h \in K_h$ and data $F := a(\cdot, w_h)$, $G := 0$, the discrete solution is $(w_h, 0)$ and the realized stability ratio is $C_h = \|w_h\|_{\mathrm{div}}/\|F\|_{V_h'}$.

datum w_h	measured growth	limit of the compensated ratio
single bubble	$C_h \sim h^{-3/2}$	$C_h\, h^{3/2} \to 3.33$
alternating bubbles	$C_h \sim h^{-2}$	$C_h\, h^2 \to 5.847$

The necessity proof gives $C_h \geq \alpha_h^{-1/2} \sim h^{-1}$; Theorem 6 gives $C_h \lesssim \alpha_h^{-1} \sim h^{-2}$. A single positive bubble pairs with the constant test function, $(\mathbf{1}, w_h)_0 \sim h$, which bounds its dual norm from below and caps the growth at $h^{-3/2}$; the alternating field is L^2-quasi-orthogonal to smooth functions and attains the order of the upper bound.

Backup: the exact one-dimensional constant

For $\Omega = (0,1)$, $V = H^1(0,1)$ with $\|\cdot\|_{\mathrm{div}}$, $Q = L^2(0,1)$, $b(\tau, v) = (\tau', v)_0$:

$$\beta = \frac{\pi}{\sqrt{1+\pi^2}} \approx 0.9528905.$$

Sketch (Exercise 2(e)): $\beta^2 = \inf_q (Tq, q)_0 / \|q\|_0^2$ with $Tq := t_q'$, where t_q Riesz-represents $b(\cdot, q)$. The eigenvalue problem $Tq = \lambda q$ has the strong form $-t'' + t = -q'$ with natural conditions $(t' - q)(0) = (t' - q)(1) = 0$; eliminating $q = t'/\lambda$ gives $t'' = -\omega^2 t$, $t'(0) = t'(1) = 0$, $\omega^2 = \lambda/(1-\lambda)$, hence $t = \cos(k\pi x)$ and $\lambda_k = k^2\pi^2/(1 + k^2\pi^2)$ with eigenfunctions $q \propto \sin(k\pi x)$, complete in $L^2(0,1)$.

The measured discrete values converge to this constant from above ($0.95345 \to 0.952893$ for $n = 8, \ldots, 128$): restricting Q_h can only increase the infimum, restricting V_h can only decrease the supremum, and the first effect dominates here.