# Stokes, stable pairs, and the solution of saddle-point systems

Mixed finite element methods — a crash course · Lecture 4 of 4

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All numerical results in these slides were produced by the verified reference implementation that accompanies the course materials.

## Where we stand after three lectures

Lecture 3 closed with a bridge: one construction — the commuting diagram — delivered $(\mathrm{InfSup}_h)$ for a whole family of pairs at once. For Stokes no such construction exists; the discrete inf-sup condition is established pair by pair.

- **Stable Stokes pairs:** one complete stability proof (MINI), the classical pairs stated with references, and the Lecture-1 failures revisited.
- **Solution algorithms** for the block systems of all four lectures (the promise of Lecture 1).
- **A map of the field:** annotated pointers beyond the two model problems.

The recurring object, for the last time:

$$\begin{pmatrix} \mathsf{A} & \mathsf{B}^T \\ \mathsf{B} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \mathbf{g} \end{pmatrix}.$$

**One verification is proved completely; the others are stated.**

# The Stokes problem, recalled

Exactly as posed in Lecture 1: find $(\boldsymbol{u}, p) \in V \times Q$, with $V = (H_0^1(\Omega))^d$ and $Q = L_0^2(\Omega)$, such that

$$\begin{aligned} \nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) - (\operatorname{div} \boldsymbol{v},\, p) &= (\boldsymbol{f}, \boldsymbol{v}) \qquad && \forall\, \boldsymbol{v} \in V, \\ -(\operatorname{div} \boldsymbol{u},\, q) &= 0 && \forall\, q \in Q. \end{aligned}$$

In the abstract template: $a(\boldsymbol{u}, \boldsymbol{v}) = \nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v})$, $b(\boldsymbol{v}, q) = -(\operatorname{div} \boldsymbol{v}, q)$, $F(\boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v})$, $G = 0$.

- Norms: $\|\nabla \boldsymbol{v}\|_0$ on $V$ (equivalent to $\|\cdot\|_1$ by the Poincaré inequality) and $\|\cdot\|_0$ on $Q$.
- $Q = L_0^2(\Omega)$: for enclosed flow the pressure is determined only up to an additive constant.
- Continuity: $\|a\| = \nu$ and $|b(\boldsymbol{v}, q)| \leq \sqrt{d}\, \|\nabla \boldsymbol{v}\|_0\, \|q\|_0$ (improved to $\|b\| \leq 1$ in Lemma 43).

The viscosity $\nu$ stays a symbol throughout; frames 26, 28–29, and 31 depend on tracking it.

## Ellipticity holds on all of $V$

The kernel of the constraint is

$$K = \{\boldsymbol{v} \in V : \; b(\boldsymbol{v}, q) = 0 \; \forall q \in Q\} = \{\boldsymbol{v} \in V : \; \operatorname{div} \boldsymbol{v} = 0\} :$$

the divergence theorem gives $\int_\Omega \operatorname{div} \boldsymbol{v} \, dx = 0$ for $\boldsymbol{v} \in (H_0^1(\Omega))^d$, so $\operatorname{div} \boldsymbol{v} \in L_0^2(\Omega)$ automatically, and testing against all of $L_0^2$ forces $\operatorname{div} \boldsymbol{v} = 0$.

For *every* $\boldsymbol{v} \in V$, not only on $K$:

$$a(\boldsymbol{v}, \boldsymbol{v}) = \nu \, \|\nabla \boldsymbol{v}\|_0^2.$$

Hypothesis $(\mathrm{ElKer})$ holds on all of $V$ with $\alpha = \nu$. This is the announced complementarity with the mixed Laplacian, where $(\mathrm{ElKer})$ held only on the kernel and $(\mathrm{InfSup})$ followed from an elementary lifting (Lecture 2); for Stokes the situation is reversed.

**Ellipticity is global and costs nothing; the inf-sup condition carries the entire difficulty of well-posedness.**

# The continuous inf-sup condition

**Theorem 30 (continuous inf-sup condition; stated).** *Let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, be a bounded Lipschitz domain. There exists $\beta = \beta(\Omega) > 0$ such that*

$$\sup_{\boldsymbol{v} \in V \setminus \{\boldsymbol{0}\}} \frac{b(\boldsymbol{v}, q)}{\|\nabla \boldsymbol{v}\|_0} \;\geq\; \beta \, \|q\|_0 \qquad \textit{for all } q \in L_0^2(\Omega).$$

Lemma 2 of Lecture 2 translates the statement (its fourth use):

- *surjectivity with bounded lifting:* for every $q \in L_0^2(\Omega)$ there exists $\boldsymbol{v} \in (H_0^1(\Omega))^d$ with $\operatorname{div} \boldsymbol{v} = q$ and $\|\nabla \boldsymbol{v}\|_0 \leq \beta^{-1} \|q\|_0$;
- *a norm equivalence (Nečas):* $\|q\|_0 \leq \beta^{-1} \sup_{\boldsymbol{v}} (q, \operatorname{div} \boldsymbol{v}) / \|\nabla \boldsymbol{v}\|_0$.

The proof is not given in this course; every known argument needs more than the Lecture-2 Poisson lifting: the Bogovskii integral operator on star-shaped domains with a covering argument, or the Nečas inequality; the constant $\beta$ depends on the domain.

*References:* Girault–Raviart, Ch. I; Bogovskii (1979); Nečas (1966).

## Well-posedness; the complementarity, completed

**Corollary 31.** *Under Theorem 30 the Stokes problem has a unique solution* $(\boldsymbol{u}, p) \in V \times Q$*, and with* $\|\boldsymbol{f}\|_{-1} := \sup_{\boldsymbol{v}} (\boldsymbol{f}, \boldsymbol{v})/\|\nabla \boldsymbol{v}\|_0$*,*

$$\|\nabla \boldsymbol{u}\|_0 \le \nu^{-1} \|\boldsymbol{f}\|_{-1}, \qquad \|p\|_0 \le 2\,\beta^{-1} \|\boldsymbol{f}\|_{-1}.$$

**Proof.** The Brezzi theorem (Lecture 2, Theorem 3) applies with $\alpha = \nu$ (frame 4) and $\beta$ from Theorem 30. Since $G = 0$, the solution satisfies $\boldsymbol{u} \in K$; testing with $\boldsymbol{v} = \boldsymbol{u}$ gives $\nu \|\nabla \boldsymbol{u}\|_0^2 = (\boldsymbol{f}, \boldsymbol{u}) \le \|\boldsymbol{f}\|_{-1} \|\nabla \boldsymbol{u}\|_0$. For the pressure, $b(\boldsymbol{v}, p) = F(\boldsymbol{v}) - a(\boldsymbol{u}, \boldsymbol{v})$ for all $\boldsymbol{v}$, so Theorem 30 gives $\beta \|p\|_0 \le \|\boldsymbol{f}\|_{-1} + \nu \|\nabla \boldsymbol{u}\|_0 \le 2 \|\boldsymbol{f}\|_{-1}$. □

The table announced in Lecture 2 (frame 15), completed:

| | mixed Laplacian | Stokes |
|---|---|---|
| (ElKer) | only on $K$; $\alpha = 1/\kappa_+$ | all of $V$; $\alpha = \nu$ |
| (InfSup) | elementary lifting | Theorem 30; stated |
| $K_h \subset K$? | yes, by design (Lecture 3) | in general no (frames 17, 26) |
| discrete inf-sup | one construction, family | pair by pair (today) |

# The discrete problem: one hypothesis is free

Choose conforming subspaces $V_h \subset V$, $Q_h \subset Q$; the Galerkin problem is the block system of frame 2.

**Observation 32.** *For every conforming pair,* $a(\boldsymbol{v}_h, \boldsymbol{v}_h) = \nu \|\nabla \boldsymbol{v}_h\|_0^2$*: hypothesis* $(\mathrm{ElKer}_h)$ *holds with* $\alpha_h = \nu$*, uniformly in* $h$ *and irrespective of* $K_h$*. By the discrete theory of Lecture 2 (Theorems 6 and 7; necessity by Theorem 10, whose symmetric positive-semidefinite case covers the Stokes forms), a conforming Stokes pair is uniformly stable if and only if* $(\mathrm{InfSup}_h)$ *holds with* $\beta_h \geq \beta_* > 0$ *independent of* $h$*.*

One the the main tools remains Fortin's criterion (Lecture 2, Theorem 11): operators $\Pi_h : V \to V_h$ with

(F1) $b(\boldsymbol{v} - \Pi_h \boldsymbol{v}, q_h) = 0 \; \forall q_h \in Q_h$, (F2) $\|\nabla \Pi_h \boldsymbol{v}\|_0 \leq C_\Pi \|\nabla \boldsymbol{v}\|_0$

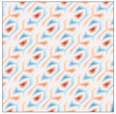
yield $\beta_h \geq \beta / C_\Pi$; the composition lemma (Lecture 2, Lemma 13) enters on frame 12, and again on frame 16.

**Ellipticity on the discrete kernel is automatic; stability is equivalent to the uniform discrete inf-sup condition.**

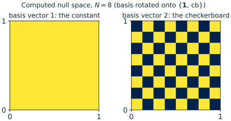
# Lecture 1 revisited: the failures, with their record

| pair | verified record | diagnosis |
|---|---|---|
| $P_1$–$P_1$ (triangles) | $\dim\ker \mathsf{B}^T = 8$ structured (either diagonal, all $N \geq 8$), $5$ perturbed; alternating connectivity $8/7$ (structured/perturbed); essential constant $0.0717 \to 0.0168$ for $N = 8 \to 40$, rate $\approx 0.97$ | $\beta_h = 0$ at every $h$: species (i); the mode-filtered pair still violates uniformity: species (ii) |
| $Q_1$–$P_0$ (quads) | two exact modes, the constant and the checkerboard; cavity $\varepsilon$-shift pressure 99.7% checkerboard, amplitude $\approx 763$ | $\ker \mathsf{B}^T \neq \{0\}$: species (i) |

The counting inequality $\dim Q_h \leq \dim V_h$ holds comfortably for both pairs; the Lecture-1 corollary — necessary, not sufficient — stands.



a $P_1$–$P_1$ kernel mode (Lecture 2)




the $Q_1$–$P_0$ kernel (Lecture 1)

## What (F1) requires for piecewise constant pressures

For $q_h \in Q_h = P_0^{\mathrm{disc}}(\mathcal{T}_h) \cap L_0^2(\Omega)$ and $\boldsymbol{w} \in (H_0^1(\Omega))^2$, the divergence theorem localizes the constraint to the element boundaries:

$$b(\boldsymbol{w}, q_h) = -\sum_{K\in\mathcal{T}_h} q_h|_K \int_K \operatorname{div}\boldsymbol{w}\,dx = -\sum_{K\in\mathcal{T}_h} q_h|_K \int_{\partial K} \boldsymbol{w}\cdot\boldsymbol{n}\,ds.$$

Condition (F1) is therefore implied by *edge*-mean conditions: $\int_e \Pi_h \boldsymbol{v}\,ds = \int_e \boldsymbol{v}\,ds$ for every edge $e$. The space $(P_2^{\mathrm{cont}})^2$ contains, for every edge, the quadratic edge function $\varphi_e$: it vanishes at all vertices and on all other edges, and $\int_e \varphi_e\,ds = \frac{2}{3}|e| \neq 0$ — one free direction per edge and per component. The correction operator, constructed on the board: $\Pi^2\boldsymbol{w} := \sum_e \frac{3}{2|e|}\left(\int_e \boldsymbol{w}\,ds\right)\varphi_e$. Board: the construction, membership in $V_h$, and (F1); the slides then prove the local bound and compose.

**For discontinuous pressures the constraint acts through element boundary fluxes; edge degrees of freedom provide the required control.**

# Quasi-interpolation (stated)

**Lemma 33 (Clément/Scott–Zhang quasi-interpolation; stated).** *There exist a linear operator $R_h : H_0^1(\Omega) \to P_1^{\mathrm{cont}}(\mathcal{T}_h) \cap H_0^1(\Omega)$, applied componentwise to vector fields, and a constant $C_R$ depending only on the shape-regularity constant $\sigma$, such that*

$$\begin{aligned} &\text{(SZ1)} \quad \|\nabla R_h v\|_0 \le C_R \, \|\nabla v\|_0, \\ &\text{(SZ2)} \quad \|v - R_h v\|_{0,K} \le C_R \, h_K \, \|\nabla v\|_{0,\omega_K} \quad \forall K \in \mathcal{T}_h, \end{aligned}$$

*where $\omega_K$ is the patch of elements touching $K$.*

No proof is given in this course; a construction sketch is on backup frame B1. Nodal interpolation is unavailable here — point values are not defined on $H^1$ for $d \ge 2$ — the same obstruction pattern as for the canonical interpolant on raw $H(\mathrm{div})$ in Lecture 3 (frame 14), resolved there by $(\mathrm{InfSup}_W)$ and here by local averaging.

*References:* Clément (1975); Scott–Zhang (1990).

# The edge correction is locally bounded

**Lemma 34 (edge correction).** *Let $\Pi^2$ be the operator of frame 9. Then:*
(i) $\int_e \Pi^2\boldsymbol{w}\,ds = \int_e \boldsymbol{w}\,ds$ *for every edge $e$; for $\boldsymbol{w} \in (H^1_0(\Omega))^2$, $\Pi^2\boldsymbol{w} \in V_h$ and $b(\boldsymbol{w} - \Pi^2\boldsymbol{w},\, q_h) = 0$ for all $q_h \in P_0^{\mathrm{disc}}$;*
(ii) *for every $K \in \mathcal{T}_h$, with $C_E = C_E(\sigma)$,*

$$\|\nabla(\Pi^2\boldsymbol{w})\|_{0,K} \le C_E\big(h_K^{-1}\|\boldsymbol{w}\|_{0,K} + \|\nabla\boldsymbol{w}\|_{0,K}\big).$$

**Proof.** (i) was verified on the board: on $e$, the trace of $\Pi^2\boldsymbol{w}$ is its coefficient times $\varphi_e$; boundary edges carry zero coefficients; the frame-9 identity gives (F1). (ii): on each $K$, with $\|\nabla\varphi_e\|_{0,K} \le C(\sigma)$ and the Cauchy–Schwarz inequality on $e$,

$$\|\nabla(\Pi^2\boldsymbol{w})\|_{0,K} \le \sum_{e\subset\partial K} \tfrac{3}{2|e|}\left|\int_e \boldsymbol{w}\,ds\right| \|\nabla\varphi_e\|_{0,K} \le C\sum_{e\subset\partial K}|e|^{-1/2}\|\boldsymbol{w}\|_{0,e}\,,$$

and the scaled trace inequality $\|\boldsymbol{w}\|_{0,e}^2 \le C\big(h_K^{-1}\|\boldsymbol{w}\|_{0,K}^2 + h_K\|\nabla\boldsymbol{w}\|_{0,K}^2\big)$ concludes. $\square$

Property (ii) is exactly the hypothesis of the composition lemma (Lecture 2, Lemma 13); the pieces are assembled on the next frame.

## $P_2$–$P_0$ is uniformly stable

**Theorem 35 ($P_2$–$P_0$ for Stokes, $d = 2$).** *On a shape-regular simplicial family, the pair* $V_h = (P_2^{\mathrm{cont}}(\mathcal{T}_h) \cap H_0^1(\Omega))^2$, $Q_h = P_0^{\mathrm{disc}}(\mathcal{T}_h) \cap L_0^2(\Omega)$ *satisfies* $(\mathrm{InfSup}_h)$ *uniformly in* $h$, *and* $\|\nabla(\boldsymbol{u} - \boldsymbol{u}_h)\|_0 + \|p - p_h\|_0 = O(h)$ *for* $\boldsymbol{u} \in H^2$, $p \in H^1$.

**Proof.** Set $\Pi_h := R_h + \Pi^2(I - R_h)$ (Lecture 2, Lemma 13, with $\Pi^1 = R_h$ from Lemma 33) and write $\boldsymbol{r} := \boldsymbol{v} - R_h\boldsymbol{v}$.

**(F1):** by Lemma 34(i), $b(\boldsymbol{v} - \Pi_h\boldsymbol{v}, q_h) = -\sum_K q_h|_K \int_{\partial K}(I - \Pi^2)\boldsymbol{r}\cdot\boldsymbol{n}\,ds = 0$.

**(F2):** by Lemma 34(ii), (SZ1), (SZ2), and the bounded overlap of patches,

$$\begin{aligned}\|\nabla\Pi_h\boldsymbol{v}\|_0 &\le C_R\|\nabla\boldsymbol{v}\|_0 + \Big(\sum_K C_E^2\big(h_K^{-1}\|\boldsymbol{r}\|_{0,K} + \|\nabla\boldsymbol{r}\|_{0,K}\big)^2\Big)^{1/2}\\ &\le \underbrace{\big(C_R + C_E(1 + C_R + C_R M^{1/2})\big)}_{=:C_\Pi}\|\nabla\boldsymbol{v}\|_0.\end{aligned}$$

Theorem 11 of Lecture 2 concludes: $\beta_h \ge \beta/C_\Pi$. The error orders follow from Corollary 39 (frame 17); the accuracy is limited by the piecewise constant pressure — the balance phenomenon of Lecture 2 (Corollary 8). □

**The same pair fails for one problem and is stable for another; design principle 1 of the course is now a proved fact.**

## Where the constraint acts

**Observation 36 (localization).** *For $\boldsymbol{w} \in (H_0^1(\Omega))^2$ and piecewise smooth $q_h$:*

$$b(\boldsymbol{w}, q_h) = \sum_{K \in \mathcal{T}_h} \int_K \boldsymbol{w} \cdot \nabla q_h \, dx \; - \sum_{e \text{ interior}} \int_e (\boldsymbol{w} \cdot \boldsymbol{n}_e) \, [\![ q_h ]\!]_e \, ds.$$

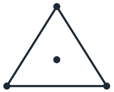
- $q_h$ continuous: volume term only; interior degrees of freedom suffice — element means when $\nabla q_h$ is piecewise constant.
- $q_h \in P_0^{\text{disc}}$: jump term only; the facet means of $\boldsymbol{w} \cdot \boldsymbol{n}$ must be controlled — and interior bubbles, having zero trace, cannot help.
- $d = 3$: $P_2$ has no face degrees of freedom; repair: one normal face bubble per face (Bernardi–Raugel, 1985).
- Facet control is necessary, not sufficient: $P_2$–$P_1^{\text{disc}}$ has edge degrees of freedom, yet is not uniformly stable (singular vertices; frame 27).
- Crouzeix–Raviart realizes the mechanism outside $H^1$ (backup B2).

**The pressure space decides where the constraint acts: interior degrees of freedom for continuous, facet for discontinuous pressures.**

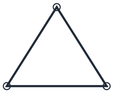
# The MINI element

On a simplicial mesh, let $b_K := \lambda_{K,1} \cdots \lambda_{K,d+1}$ denote the interior bubble of $K$ ($b_K \in H^1_0(K)$), and $B(\mathcal{T}_h) := \operatorname{span}\{b_K : K \in \mathcal{T}_h\}$.
The MINI pair (Arnold–Brezzi–Fortin, 1984):

$$V_h := \left(P_1^{\text{cont}}(\mathcal{T}_h) \cap H^1_0(\Omega) \oplus B(\mathcal{T}_h)\right)^d, \quad Q_h := P_1^{\text{cont}}(\mathcal{T}_h) \cap L^2_0(\Omega).$$



velocity: $P_1 \oplus B$, each component



pressure: $P_1^{\text{cont}}$

In two dimensions, $\dim V_h = 2(\#V_{\text{int}} + \#T)$ and $\dim Q_h = \#V - 1$: the counting inequality holds with a wide margin. By Observation 36, (F1) for $Q_h = P_1^{\text{cont}}$ reduces to $d$ mean values per element; the bubbles supply exactly $d$ free directions per element.

**The failing $P_1$–$P_1$ pair is repaired by one bubble per element and component; the name abbreviates this minimality.**

# The bubble correction

**Lemma 37.** *Define* $\Pi_h^B : (L^2(\Omega))^d \to (B(\mathcal{T}_h))^d$ *elementwise by*

$$\Pi_h^B \boldsymbol{w}\big|_K := \boldsymbol{c}_K\, b_K, \qquad \boldsymbol{c}_K := \frac{\int_K \boldsymbol{w}\,dx}{\int_K b_K\,dx} \in \mathbb{R}^d.$$

*Then* $\Pi_h^B$ *maps into* $(B(\mathcal{T}_h))^d \subset V_h \cap (H_0^1(\Omega))^d$*, and:*
(i) $\int_K \Pi_h^B \boldsymbol{w}\,dx = \int_K \boldsymbol{w}\,dx$ *for every* $K \in \mathcal{T}_h$*;*
(ii) *for* $d = 2$*,* $\|\nabla(\Pi_h^B \boldsymbol{w})\|_{0,K} \le C_B\, h_K^{-1} \|\boldsymbol{w}\|_{0,K}$ *with* $C_B = \frac{45}{2}\,\sigma$*.*

**Proof.** (i) is immediate. (ii): the formula $\int_K \lambda_1^{a_1}\lambda_2^{a_2}\lambda_3^{a_3}\,dx = |K|\,\frac{2!\,a_1!a_2!a_3!}{(a_1+a_2+a_3+2)!}$ gives $\int_K b_K = |K|/60$, hence $|\boldsymbol{c}_K| \le 60\,|K|^{-1}\big|\int_K \boldsymbol{w}\,dx\big| \le 60\,|K|^{-1/2}\|\boldsymbol{w}\|_{0,K}$ by the Cauchy–Schwarz inequality. Moreover, with $|\nabla\lambda_i| = |e_i|/(2|K|)$ and $\lambda_j\lambda_k \le \frac{1}{4}$, the product rule gives $|\nabla b_K| \le 3h_K/(8|K|)$ and $\|\nabla b_K\|_{0,K} \le \frac{3h_K}{8}\,|K|^{-1/2}$. The rows of $\nabla(\boldsymbol{c}_K b_K)$ are $(\boldsymbol{c}_K)_i \nabla b_K$, hence $\|\nabla(\Pi_h^B \boldsymbol{w})\|_{0,K} = |\boldsymbol{c}_K|\,\|\nabla b_K\|_{0,K} \le \frac{45}{2}\,\frac{h_K}{|K|}\,\|\boldsymbol{w}\|_{0,K}$, and $|K| = \rho_K s_K > \rho_K h_K \ge h_K^2/\sigma$ (inradius times semiperimeter; the perimeter exceeds $2h_K$). $\square$

Three dimensions: $\int_K b_K = |K|/840$; identical argument, modified constants.

# MINI is stable

**Theorem 38.** *Let $\{\mathcal{T}_h\}$ be shape-regular, $d \in \{2,3\}$. The MINI pair satisfies* $(\mathrm{InfSup}_h)$ *with* $\beta_h \geq \beta_* := \beta/\big(C_R(1 + C_B M^{1/2})\big) > 0$, *with $\beta$ from Theorem 30, $C_R, C_B$ from Lemmas 33 and 37, and $M = M(\sigma)$ the patch-overlap bound.*

**Proof.** Set $\Pi_h := R_h + \Pi_h^B(I - R_h) : V \to V_h$ (Lecture 2, Lemma 13, with $\Pi^1 = R_h$, $\Pi^2 = \Pi_h^B$; components vanish on $\partial\Omega$).
**(F1):** by Observation 36 and Lemma 37(i), $b(\boldsymbol{v} - \Pi_h \boldsymbol{v},\, q_h) = \sum_K \nabla q_h|_K \cdot \int_K (I - \Pi_h^B)(\boldsymbol{v} - R_h \boldsymbol{v})\, dx = 0$.
**(F2):** by Lemma 37(ii), (SZ2), and the bounded overlap of patches,

$$
\begin{aligned}
\|\nabla \Pi_h \boldsymbol{v}\|_0 &\leq C_R \|\nabla \boldsymbol{v}\|_0 + \Big(\sum_K C_B^2 h_K^{-2} \|\boldsymbol{v} - R_h \boldsymbol{v}\|_{0,K}^2\Big)^{1/2} \\
&\leq C_R\big(1 + C_B M^{1/2}\big) \|\nabla \boldsymbol{v}\|_0.
\end{aligned}
$$

Theorem 11 of Lecture 2 concludes: $\beta_h \geq \beta / C_\Pi$ with $C_\Pi = C_R(1 + C_B M^{1/2})$. □

# Convergence of MINI

**Corollary 39.** *Assume* $(\mathrm{InfSup}_h)$ *(Theorems 35, 38, or 42) and let* $\boldsymbol{u} \in (H^2(\Omega))^d$, $p \in H^1(\Omega)$. *Then, with* $C = C(\beta_*)$,

$$\|\nabla(\boldsymbol{u}-\boldsymbol{u}_h)\|_0 \le C\Big(\inf_{\boldsymbol{v}_h \in V_h} \|\nabla(\boldsymbol{u}-\boldsymbol{v}_h)\|_0 + \nu^{-1} \inf_{q_h \in Q_h} \|p-q_h\|_0\Big),$$

$$\|p-p_h\|_0 \le C\Big(\inf_{q_h \in Q_h} \|p-q_h\|_0 + \nu\,\|\nabla(\boldsymbol{u}-\boldsymbol{u}_h)\|_0\Big),$$

*and first-order interpolation gives*

$\|\nabla(\boldsymbol{u}-\boldsymbol{u}_h)\|_0 = O(h)\,(|\boldsymbol{u}|_2 + \nu^{-1}|p|_1)$, $\|p-p_h\|_0 = O(h)$.

**Proof.** Theorem 7 of Lecture 2, instantiated with $\alpha_h = \|a\| = \nu$ and $\|b\| \le 1$ (Lemma 43, frame 26; the elementary bound $\sqrt{d}$ suffices here). □

- The regularity holds on convex polygons (Kellogg–Osborn, 1976); by a duality argument, $\|\boldsymbol{u}-\boldsymbol{u}_h\|_0 = O(h^2)$ on convex domains (stated; Girault–Raviart).
- The term $\nu^{-1}\inf_{q_h}\|p-q_h\|_0$ is not an artifact of the proof: pairs with $K_h \subset K$ decouple the velocity error from the pressure (Lecture 2, Corollary 8; Lecture 3), while for MINI and Hood–Taylor the pollution is real (frame 26: *pressure robustness*).

## Hood–Taylor: the last secret

Lecture 1, Exercise 1(e), established that the one-dimensional $P_2$–$P_1$ pair is solvable at every fixed $h$ and deferred its stability. The lowest-order **Hood–Taylor** pair (Taylor & Hood, 1973; the element predates the theory):

$$V_h = (P_2^{\text{cont}}(\mathcal{T}_h) \cap H_0^1(\Omega))^2, \qquad Q_h = P_1^{\text{cont}}(\mathcal{T}_h) \cap L_0^2(\Omega).$$

By Observation 36, condition (F1) again reduces to element mean values. However, a quadratic polynomial vanishing on all three edges of a triangle vanishes identically: $(P_2^{\text{cont}})^2$ contains no function supported in a single element. The bubble correction is unavailable, and the edge correction of Theorem 35 controls boundary fluxes, not element means.

**A third mechanism is required, acting on patches of elements rather than on single elements or single facets.**

# The macroelement technique

**Theorem 40 (macroelement technique, two-level form; after Stenberg; stated).** *Let* $\{\mathcal{M}_h\}$ *be partitions of* $\{\mathcal{T}_h\}$ *into macroelements (connected unions of elements) such that:*

(M0) *each macroelement consists of at most* $L$ *elements, and the family is shape regular;*
(M1) *for every* $M \in \mathcal{M}_h$,
$N_M := \{q \in Q_h|_M : b(\boldsymbol{v}, q) = 0 \ \forall \boldsymbol{v} \in V_h \cap (H_0^1(M))^d\} = \mathbb{R}$;
(M2) *pair* $\big(V_h,\ P_0^{\mathrm{disc}}(\mathcal{M}_h) \cap L_0^2(\Omega)\big)$ *satisfies* $(\mathrm{InfSup}_h)$ *uniformly in* $h$.

*Then* $(V_h, Q_h)$ *satisfies* $(\mathrm{InfSup}_h)$ *uniformly, with a constant depending only on* $L$*, the shape regularity, and the constants in* (M1)–(M2)*.*

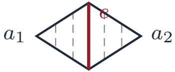
Structure of the proof (not carried out): split $q_h = \bar{q} + \tilde{q}$ into macroelement means and remainders. (M1) yields local fields $\boldsymbol{v}_M \in V_h \cap (H_0^1(M))^d$ controlling $\tilde{q}$; their uniformity over shape-regular configurations is a compactness argument — the one step this course does not prove. The splitting decouples: $b(\boldsymbol{v}_M, \bar{q}) = -\,\bar{q}|_M \int_M \operatorname{div} \boldsymbol{v}_M \, dx = 0$, so the local fields do not disturb the coarse part; (M2) controls $\bar{q}$; a weighted combination concludes.

*References:* Stenberg (1984, 1990); Brezzi–Falk (1991).

# The local condition, proved on the board

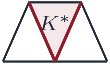
**Lemma 41 (local condition).** *Let $M$ be an edge-connected union of at least three triangles. Then, for the Hood–Taylor pair with $k = 1$, $N_M = \mathbb{R}$.*

**Proof** (board). Some triangle $K^*$ of $M$ has two edges interior to $M$ (a connected graph on $\geq 3$ vertices has a vertex of degree $\geq 2$). Testing with $\varphi_e \boldsymbol{a}$ across an interior edge $e$ gives $|K_1| \, \nabla q|_{K_1} + |K_2| \, \nabla q|_{K_2} = \mathbf{0}$ $(*)$; tangential continuity of $q$ across $e$ then gives $\nabla q \cdot \boldsymbol{t}_e = 0$ on both sides. The two non-parallel interior edges of $K^*$ force $\nabla q|_{K^*} = \mathbf{0}$; $(*)$ propagates across interior edges; continuity concludes. □




two elements:

$N_M = \mathbb{R} \oplus \operatorname{span}\{\lambda_{a_1} + \lambda_{a_2}\}$




a triangle with two interior edges:

$N_M = \mathbb{R}$

The two-element computation characterizes the kernel exactly: the surviving mode vanishes on $e$ and takes equal values at the opposite vertices (Exercise 2(a)).

# Hood–Taylor, proved

**Theorem 42 (Hood–Taylor).** *Let $d \in \{2,3\}$, $k \geq 1$, and let $\{\mathcal{T}_h\}$ be a shape-regular simplicial family such that, for $d = 2$, $\mathcal{T}_h$ contains at least three triangles, and, for $d = 3$, every tetrahedron has at least one vertex in the interior of $\Omega$. Then the pair $(P_{k+1}^{\mathrm{cont}})^d$–$P_k^{\mathrm{cont}}$ (spaces as on frame 18) satisfies $(\mathrm{InfSup}_h)$ uniformly in $h$; for $k = 1$, $\|\nabla(\boldsymbol{u} - \boldsymbol{u}_h)\|_0 + \|p - p_h\|_0 = O(h^2)$, $\|\boldsymbol{u} - \boldsymbol{u}_h\|_0 = O(h^3)$.*

**Proof** (delivered for $d = 2$, $k = 1$). Partition $\mathcal{T}_h$ into edge-connected macroelements of three to $L$ triangles (existence for every connected triangulation with at least three triangles, $L = 7$: Exercise 2(e*)). (M1) holds by Lemma 41; (M2) holds with the operator of Theorem 35: its edge-mean property gives (F1) for $P_0^{\mathrm{disc}}(\mathcal{M}_h)$ on every union of elements, with the same constant. Theorem 40 applies. Delivered: the two-dimensional condition for $k = 1$, up to the compactness step; $k \geq 2$ and $d = 3$ are cited. □

In three dimensions the local ingredient is the edge computation at an interior vertex (Exercise 2(d*)). Pressure continuity is essential: $P_2$–$P_1^{\mathrm{disc}}$ is *not* uniformly stable (singular vertices; frame 27).

*References:* Taylor & Hood (1973); Bercovier–Pironneau (1979); Verfürth (1984); Stenberg (1984, 1990); Brezzi–Falk (1991); Boffi (1994, 1997).

## Demonstration 1a: convergence rates

Manufactured solution on $\Omega = (0,1)^2$, $\nu = 1$: $\boldsymbol{u} = \mathbf{curl}\,\psi$ with $\psi = x^2(1-x)^2y^2(1-y)^2$ (exactly divergence-free, zero boundary values), $p = x^3y^3 - \frac{1}{16} \in L^2_0$; meshes $N = 8$ to $64$; the table shows the measured rates at the last refinement.

| pair | $\|\nabla(\boldsymbol{u}-\boldsymbol{u}_h)\|_0$ | $\|p-p_h\|_0$ | $\|\boldsymbol{u}-\boldsymbol{u}_h\|_0$ |
|---|---|---|---|
| $P_2$–$P_0$ | $0.97$ | $1.01$ | $1.94$ |
| MINI | $1.01$ | $1.53$ | $2.01$ |
| Hood–Taylor | $2.00$ | $2.00$ | $3.00$ |

Every row of this table is a statement from today: the $P_2$–$P_0$ row is Theorem 35, the MINI row is Corollary 39 (with the duality remark), the Hood–Taylor row is Theorem 42.

▶ **live demonstration:** notebook `L4A`, Demo 1a — convergence rates, both pairs.

The MINI pressure exceeds the guaranteed first order: $1.53$ on structured, $1.27$ on perturbed meshes; Corollary 39 is a bound from below.

## Demonstration 1b: the pencil, fourth appearance

The generalized eigenvalue problem

$$\mathsf{B}\,\mathsf{K}_V^{-1}\mathsf{B}^T\,\mathbf{q} = \lambda\,\mathsf{M}_p\,\mathbf{q} \qquad \text{on } \{\mathbf{q} \perp \mathbf{1}\},$$

introduced in Lecture 2, Exercise 1(a); measured for the Stokes pairs ($\beta_h = \sqrt{\lambda_{\min}}$):

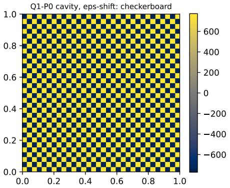
| pair | $\beta_h$ $(N = 8, 16, 24)$ | kernel count |
|---|---|---|
| MINI | $0.3143,\ 0.3136,\ 0.3134$ | $0$ |
| Hood–Taylor | $0.3662,\ 0.3656,\ 0.3654$ | $0$ |
| $P_1$–$P_1$ | $0$ | $7$ beyond the constant |

The $\varepsilon$-shifted $P_1$–$P_1$ solve does not blow up — interior forcing cannot excite the kernel at any $\varepsilon$ — but the pressure stagnates at $\|p - p_h\|_0 \approx 8 \cdot 10^{-2}$ while the velocity converges $(1.01,\ 2.01)$; a kernel-incompatible constraint datum reproduces the Lecture-1 blow-up at order $10^8$ ($|g_k| = 10^{-3}$, $\varepsilon = 10^{-8}$). Frame 28 gives this eigenvalue problem a second job.

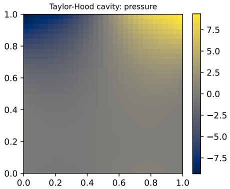
▶ **live demonstration:** notebook `L4A`, Demos 1b/1c — the pencil; the $\varepsilon$-shift.

# Demonstration 2: the cavity, revisited

The lid-driven cavity of Lecture 1, in the identical configuration (mesh, lid datum, plotting pipeline of `L1B`), discretized with Hood–Taylor.




$Q_1$–$P_0$ (Lecture 1): checkerboard, amplitude $\approx 763$



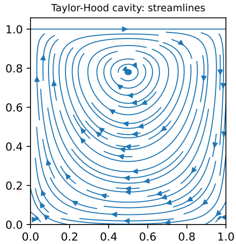

Hood–Taylor (today): pressure in $[-9.86, 9.86]$

What changed between the two pictures is the pair and nothing else.

On the quadrilateral `L1B` mesh the pair is $Q_2$–$Q_1$, uniformly stable (Boffi–Brezzi–Fortin); Theorem 42 covers the simplicial case.

▶ **live demonstration:** notebook `L4B`, Demo 2 — the cavity with Hood–Taylor.

# Demonstration 2: streamlines; the moral, retired




streamlines and velocity magnitude, Hood–Taylor cavity

The Lecture-1 moral is retired. In Lecture 1 each field had to be inspected separately because one of them was garbage; today both fields are inspected separately and both are correct.

Add-back 6 reruns the $Q_1$–$P_0$ cavity live for a side-by-side comparison (backup frame B6).

# Pressure robustness: the $\nu^{-1}$ term, quantified

**Lemma 43.** *For every $\boldsymbol{v} \in (H_0^1(\Omega))^d$, $\|\operatorname{div} \boldsymbol{v}\|_0 \le \|\nabla \boldsymbol{v}\|_0$; in particular $\|b\| \le 1$.* **Proof.** For smooth $\boldsymbol{v}$, two integrations by parts give $\|\operatorname{div} \boldsymbol{v}\|_0^2 = \int \nabla \boldsymbol{v} : (\nabla \boldsymbol{v})^T dx \le \|\nabla \boldsymbol{v}\|_0^2$; density concludes. □

**Proposition 44 (gradient forcing).** *Let $\phi \in H^1(\Omega) \cap L_0^2(\Omega)$ and $\boldsymbol{f} = \nabla \phi$. Then $(\boldsymbol{u}, p) = (\boldsymbol{0}, \phi)$, and for every pair satisfying $(\mathrm{InfSup}_h)$, $\|\nabla \boldsymbol{u}_h\|_0 \le \nu^{-1} \inf_{q_h \in Q_h} \|\phi - q_h\|_0$.*

**Proof.** $\boldsymbol{u}_h \in K_h$; for $\boldsymbol{v}_h \in K_h$ and any $q_h \in Q_h$,

$\nu(\nabla \boldsymbol{u}_h, \nabla \boldsymbol{v}_h) = (\nabla \phi, \boldsymbol{v}_h) = -(\phi, \operatorname{div} \boldsymbol{v}_h) = -(\phi - q_h, \operatorname{div} \boldsymbol{v}_h)$, since

$(\operatorname{div} \boldsymbol{v}_h, q_h) = 0$ on $K_h$. Take $\boldsymbol{v}_h = \boldsymbol{u}_h$ and apply Lemma 43. □

Measured pollution $\|\nabla \boldsymbol{u}_h\|_0$; $\phi = p^*$, $\nu = 1$, perturbed meshes:

| $N$ | MINI | Hood–Taylor | $\inf_{q_h} \|\phi - q_h\|_0$ |
|---|---|---|---|
| 16 | $4.95 \cdot 10^{-4}$ | $2.27 \cdot 10^{-4}$ | $7.43 \cdot 10^{-4}$ |
| 32 | $1.31 \cdot 10^{-4}$ | $6.94 \cdot 10^{-5}$ | $1.92 \cdot 10^{-4}$ |
| 64 | $3.36 \cdot 10^{-5}$ | $1.84 \cdot 10^{-5}$ | $4.85 \cdot 10^{-5}$ |

All columns converge with order two; the scaling in $\nu$ is exact by linearity.

**Whether $K_h \subset K$ holds is a design property with $O(\nu^{-1})$ consequences.**

## Pressure robustness: divergence-free pairs

Pairs with $K_h \subset K$ give $\boldsymbol{u}_h = \mathbf{0}$ under gradient forcing (Lecture 2, Corollary 8; realized in Lecture 3). Which pairs achieve this for Stokes, and at what cost?

- Routes to divergence-free or pressure-robust Stokes elements: Scott–Vogelius pairs $P_k$–$P_{k-1}^{\mathrm{disc}}$ ($k \geq 4$ in 2D away from singular vertices; lower $k$ on split meshes); $H(\mathrm{div})$-conforming velocities with DG couplings — the spaces of Lecture 3 as Stokes velocity spaces; velocity reconstruction in the load, built from the Lecture-3 interpolants (Linke, 2014).
- The structural statement is the **Stokes complex**: one derivative above the de Rham complex of Lecture 3, frame 16; discrete subcomplexes with the same exactness produce divergence-free stable pairs (Falk–Neilan, 2013).

*References:* Scott–Vogelius (1985); Linke (2014); Falk–Neilan (2013); John–Linke–Merdon–Neilan–Rebholz, SIAM Review (2017).

# Solving mixed systems: MINRES, preconditioned

Every discretization in this course produced the block system of frame 2, $\mathsf{K} = \left(\begin{smallmatrix} \mathsf{A} & \mathsf{B}^T \\ \mathsf{B} & 0 \end{smallmatrix}\right)$, symmetric indefinite — conjugate gradients are inadmissible, **MINRES** (Paige–Saunders, 1975) applies, its convergence governed by the preconditioned spectrum (direct solvers remain adequate at course sizes). The Schur complement $\mathsf{S} := \mathsf{B}\mathsf{A}^{-1}\mathsf{B}^T$ (eliminate $\mathbf{u}$) is SPD on zero-mean pressures exactly when $(\mathrm{InfSup}_h)$ holds; the Uzawa iteration is a gradient method on it. The ideal $\mathrm{diag}(\mathsf{A}, \mathsf{S})$ is exact in three iterations but unaffordable (Murphy–Golub–Wathen, 2000); the substitute: $\mathsf{P} = \mathrm{diag}\big(\mathsf{A},\ \nu^{-1}\mathsf{M}_p\big)$.

**Theorem 45 (spectral equivalence; stated).** *For a conforming pair with discrete inf-sup constant $\beta_h > 0$, $\beta_h^2 \le \nu\,(\mathbf{q}^T\mathsf{S}\,\mathbf{q})/(\mathbf{q}^T\mathsf{M}_p\,\mathbf{q}) \le 1$ for all $q_h \in Q_h \cap L_0^2(\Omega)$, $q_h \neq 0$.*

Upper: Lemma 43; lower: $(\mathrm{InfSup}_h)$; proof: backup B3, Exercise 3(a).

Measured at $N = 16$: MINI $[0.0983,\, 0.9851]$, Hood–Taylor $[0.1336,\, 1.0000]$.

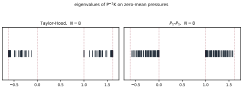
**The eigenvalue problem that measured $\beta_h$ in every lecture is, verbatim, the analysis of the preconditioner.**

# The preconditioned spectrum

**Corollary 46 (stated).** *Let* $\beta_h > 0$ *and* $\mathsf{P} = \operatorname{diag}(\mathsf{A}, \nu^{-1}\mathsf{M}_p)$*. On zero-mean pressures, every eigenvalue of* $\mathsf{P}^{-1}\mathsf{K}$ *lies in*

$$\left[\tfrac{1-\sqrt{5}}{2},\ \tfrac{1-\sqrt{1+4\beta_h^2}}{2}\right]\ \cup\ \{1\}\ \cup\ \left[\tfrac{1+\sqrt{1+4\beta_h^2}}{2},\ \tfrac{1+\sqrt{5}}{2}\right].$$

Proof via $\nu\mathsf{M}_p^{-1}\mathsf{S}\,\mathbf{q} = \lambda(\lambda-1)\mathbf{q}$ and Theorem 45: backup B4; Exercise 3(d*).




$N = 8$: Hood–Taylor fills $[-0.6179, -0.1198] \cup \{1\} \cup [1.1198, 1.6179]$ (Cor. 46, $\beta_h^2{=}0.134$); $P_1$–$P_1$ closes the origin gap to $\pm 0.005$ — the spectrum diagnoses stability.

MINRES, exact blocks, Hood–Taylor, tolerance $10^{-8}$ in the $\mathsf{P}^{-1}$-norm: $N = 16, 32, 64$: 29, 27, 27 iterations, the perturbed family identical — flat, as for the Lecture-3 trace system (live rerun: `L4B`, self-study).

*References:* Elman–Silvester–Wathen (2014); Benzi–Golub–Liesen (2005); production blocks (multigrid, mass-matrix sweep): Silvester–Wathen (1994).

# Outlook (i): elasticity and volumetric locking

Nearly incompressible linear elasticity in pure displacement form:

$$-\operatorname{div}\,\sigma(\boldsymbol{u}) = \boldsymbol{f}, \qquad \sigma(\boldsymbol{u}) = \mu\big(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T\big) + \lambda\,(\operatorname{div}\boldsymbol{u})\,\mathsf{I},$$

with Lamé parameters $\mu > 0$ and $\lambda > 0$.

In the incompressible limit $\lambda \to \infty$, low-order displacement elements *lock*: the error bound degrades with $\lambda$, and the computed deformation is spuriously stiff. This is the phenomenon announced in Lecture 1 (frame 7); the promise recorded there is redeemed on the next frame.

**The cure is a reformulation into the saddle-point structure of this course.**

# Outlook (ii): the Herrmann reformulation

Introducing $p := -\lambda\, \operatorname{div} \boldsymbol{u}$ yields: find $(\boldsymbol{u}, p)$ such that

$$\begin{aligned} \mu\big(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T,\, \nabla \boldsymbol{v}\big) - (\operatorname{div} \boldsymbol{v},\, p) &= (\boldsymbol{f}, \boldsymbol{v}), \\ -(\operatorname{div} \boldsymbol{u},\, q) - \tfrac{1}{\lambda}\,(p, q) &= 0, \end{aligned}$$

a *perturbed* Stokes system; every stable Stokes pair of this lecture yields $\lambda$-uniform error bounds. The required perturbed saddle-point theory is a modest extension of Lecture 2 and is not developed in this course.

Mixed elasticity proper takes the stress as the primary unknown and enforces its symmetry weakly; the resulting elements are a field of their own.

*References:* Herrmann (1965); Boffi–Brezzi–Fortin (2013); Arnold–Winther (2002); Arnold–Falk–Winther (2007).

## Outlook (iii): stabilized equal-order pairs

- The $P_1$–$P_1$ pair can be rescued by modifying the equations. Brezzi–Pitkäranta add

$$- \gamma \sum_{K \in \mathcal{T}_h} h_K^2 \, (\nabla p_h, \nabla q_h)_K ,$$

  which removes $\ker \mathsf{B}^T$ and restores a uniform stability estimate; consistent variants (PSPG and relatives) avoid the accuracy loss of the plain penalty.

- The Lecture-1 $\varepsilon$-shift, reread: it was an *inconsistent* regularization, and the measured amplitudes $3.8 \cdot 10^5$ (1D) and $763$ (2D) were the symptom. Correct stabilization perturbs the equations consistently and by a mesh-dependent amount.

*References:* Brezzi–Pitkäranta (1984); Hughes–Franca–Balestra (1986).

# Outlook (iv): Maxwell, Nédélec, FEEC

- Lecture 3, frame 16, named the three-dimensional column: $H(\mathrm{curl})$ and the Nédélec elements, with tangential traces in the role that normal traces played for $H(\mathrm{div})$.
- The finite element exterior calculus organizes these constructions — and the commuting-diagram proof pattern of Lecture 3 — into one framework. Time-harmonic Maxwell problems carry the mixed structure with their own subtleties.

*References:* Arnold–Falk–Winther (2006); Arnold (2018); Monk (2003).

# Outlook (v): a posteriori estimation; DG and HDG

- Residual-based a posteriori estimators exist for both model problems of this course, with reliability and efficiency constants that again involve the inf-sup machinery.
- Discontinuous Galerkin methods develop the broken-space idea of Crouzeix–Raviart systematically. Hybridizable DG generalizes the construction of Lecture 3 — broken spaces, facet multipliers, SPD condensed systems — into a design principle.

*References:* Verfürth (1989; 2013); Cockburn–Gopalakrishnan–Lazarov (2009).

# Outlook (vi): eigenvalue problems — the bridge

- The source problems of this course are under control. The corresponding eigenvalue problems do not follow automatically: uniform $(\mathrm{ElKer}_h)$ and $(\mathrm{InfSup}_h)$ do *not* guarantee spurious-free eigenvalue approximation for mixed problems.
- Pairs exist that converge for every source problem and still produce spurious eigenvalues; the correct conditions are of a different nature (discrete compactness; the Boffi–Brezzi–Gastaldi conditions).
- This topic is developed in detail in the lecture about eigenvalue approximation.

*References:* Boffi–Brezzi–Gastaldi (1997, 2000); Boffi, Acta Numerica (2010).

# The contract, discharged

**Q1** (Lecture 1, frame 32)**.** When is the continuous saddle-point problem well-posed?
*Answered* by the Brezzi theorem (Lecture 2, Theorem 3).
**Q2.** When is that inherited uniformly in $h$?
*Answered* by the discrete theory (Lecture 2), constructively in Lecture 3 and pair by pair today.

| Lecture-1 experiment | diagnosis (Lecture 2) | resolution |
|---|---|---|
| $P_1$–$P_0$ works; flatline $4.099$ | $\beta_h \approx 0.953$, uniform | the pair was $RT_0$–$P_0$ (Lecture 3) |
| $P_1$–$P_1$ singular; mode $\chi_h$ | $\beta_h = 0$: species (i) | MINI (Thm. 38); stabilization (frame 32) |
| $P_2$–$P_0$ wrong limits $0.9129$, $0.8333$ | $\alpha_h = h^2/60$: species (ii) | channel closed for Stokes (Obs. 32); proved stable (Thm. 35) |
| $Q_1$–$P_0$ checkerboard, $763$ | two exact modes | Hood–Taylor (Thm. 42; Demo 2) |

The block matrix of frame 2 has been the same object in all four lectures; solvability, stability, elements, and solvers all attach to the constants $\alpha$ and $\beta$.

# Exercises and reading

**Exercise sheet 4**: 1. bubble constants and the operator of Lemma 37; 2. Hood–Taylor and the macroelement technique (two-triangle kernel, three elements, condition (M2), $d = 3$, partitions); 3. solver theory (Theorem 45, spectral interval, preconditioned eigenvalues); 4. measurements (iteration counts, $\nu$-sweep, $\varepsilon$-shift, gradient forcing).

**Reading.**

- Boffi–Brezzi–Fortin, *Mixed Finite Element Methods and Applications*, 2013 — everything in this course and beyond.
- Gatica, *A Simple Introduction to the Mixed Finite Element Method*, 2014 — the gentle companion.
- Girault–Raviart, *Finite Element Methods for Navier–Stokes Equations*, 1986 — the classical Stokes reference.
- Elman–Silvester–Wathen, *Finite Elements and Fast Iterative Solvers*, 2nd ed., 2014 — the solver section, in depth.

Every experiment of Lecture 1 now has a theorem, a constant, and a cure attached to it.

# Backup frames

## B1 — Quasi-interpolation: construction sketch

For each interior vertex $z$ of $\mathcal{T}_h$, define $(R_h v)(z)$ as an average of $v$ near $z$: over the vertex patch (Clément) or over one fixed element or face containing $z$ (Scott–Zhang); set $(R_h v)(z) = 0$ at boundary vertices, which places $R_h v$ in $H^1_0(\Omega)$.

- (SZ1) follows from the $H^1$-stability of the local averages, an inverse estimate on each element, and the finite overlap of the patches.
- (SZ2) follows from the Poincaré inequality on patches: the average reproduces constants, so $v - R_h v$ is controlled by $h_K \|\nabla v\|_{0,\omega_K}$ locally.
- The Scott–Zhang variant reproduces piecewise affine boundary data exactly; for the homogeneous condition used in this lecture the two variants serve equally.

No proof is claimed on this frame; the references of frame 10 contain the complete arguments.

## B2 — Crouzeix–Raviart: a nonconforming alternative

**Theorem (Crouzeix–Raviart, 1973; stated).** *Let $CR_1(\mathcal{T}_h)$ be the space of piecewise affine functions whose means are continuous across interior edges, $\int_e [\![v_h]\!]\, ds = 0$ (equivalently: continuity at edge midpoints), and vanish in the mean on boundary edges. With $V_h = (CR_1(\mathcal{T}_h))^d \not\subset V$, $Q_h = P_0^{\mathrm{disc}}(\mathcal{T}_h) \cap L_0^2(\Omega)$, and the broken gradient norm $\|\nabla_h \cdot\|_0$, the pair satisfies the discrete inf-sup condition uniformly, and the method converges with order one in the broken norm.*

- The interpolation operator fixing edge means satisfies $\nabla_h(I_{CR}\boldsymbol{v})|_K = \frac{1}{|K|}\int_K \nabla \boldsymbol{v}\, dx$: a Fortin-type operator with constant exactly $1$ in the broken norm.
- The price is conformity: Galerkin orthogonality fails, and a consistency error must be estimated (second Strang lemma).
- The gain: $\mathrm{div}_h\, \boldsymbol{u}_h = 0$ holds elementwise and exactly — the broken analogue of $K_h \subset K$ (pressure robustness, frames 26–27).

**Two distinct repairs of the same failure: enrichment of the velocity space, or relaxation of its continuity, each at a different price.**

# B3 — Proof of Theorem 45 (spectral equivalence)

**Theorem 45 (spectral equivalence; stated on frame 28).** *For a conforming pair with discrete inf-sup constant* $\beta_h > 0$, $\beta_h^2 \le \nu\,(\mathbf{q}^T \mathsf{S}\,\mathbf{q})/(\mathbf{q}^T \mathsf{M}_p\,\mathbf{q}) \le 1$ *for all* $q_h \in Q_h \cap L_0^2(\Omega)$, $q_h \neq 0$. **Proof.** Since A is SPD, the Cauchy–Schwarz inequality in the A-inner product (equality at $\mathbf{v} = \mathsf{A}^{-1}\mathsf{B}^T\mathbf{q}$) gives

$$\mathbf{q}^T \mathsf{S}\mathbf{q} = \max_{\mathbf{v}\neq\mathbf{0}} \frac{(\mathbf{q}^T \mathsf{B}\mathbf{v})^2}{\mathbf{v}^T \mathsf{A}\mathbf{v}} = \frac{1}{\nu} \max_{\boldsymbol{v}_h \in V_h \setminus \{\mathbf{0}\}} \frac{b(\boldsymbol{v}_h, q_h)^2}{\|\nabla \boldsymbol{v}_h\|_0^2}.$$

The lower bound is $(\mathrm{InfSup}_h)$; the upper follows from Lemma 43 and $\mathbf{q}^T \mathsf{M}_p \mathbf{q} = \|q_h\|_0^2$. $\square$

Exercise 3(a) reproduces this argument; Exercise 3(c) verifies the interval numerically.

# B4 — Proof of Corollary 46 (eigenvalue inclusion)

The statement is on frame 29. **Proof.** The eigenproblem reads $\mathsf{A}\mathbf{u} + \mathsf{B}^T\mathbf{q} = \lambda\mathsf{A}\mathbf{u},\;\; \mathsf{B}\mathbf{u} = \lambda\,\nu^{-1}\mathsf{M}_p\mathbf{q}$. If $\mathbf{q} = \mathbf{0}$: $\lambda = 1$ and $\mathbf{u} \in \ker \mathsf{B}$. If $\mathbf{q} \neq \mathbf{0}$: $\lambda \neq 1$ (otherwise $\mathsf{B}^T\mathbf{q} = \mathbf{0}$, excluded on zero-mean pressures by $\beta_h > 0$), so $\mathbf{u} = (\lambda - 1)^{-1}\mathsf{A}^{-1}\mathsf{B}^T\mathbf{q}$, and substitution gives $\nu\,\mathsf{M}_p^{-1}\mathsf{S}\,\mathbf{q} = \lambda(\lambda - 1)\,\mathbf{q}$. By Theorem 45, $\sigma := \lambda(\lambda - 1) \in [\beta_h^2, 1]$, and $\lambda = \left(1 \pm \sqrt{1 + 4\sigma}\right)/2$ is monotone in $\sigma$ on each branch. $\square$

Exercise 3(d*) derives the inclusion from $\lambda(\lambda - 1) = \sigma$ and verifies $\{1, (1 \pm \sqrt{5})/2\}$ with the exact Schur block (Murphy–Golub–Wathen, 2000).

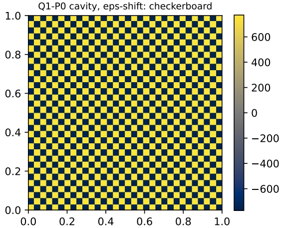
## B5 — $\nu$-robustness of the iteration counts

MINRES iterations at $N = 32$, Hood–Taylor, exact blocks; residuals in the $\mathsf{P}^{-1}$-norm:

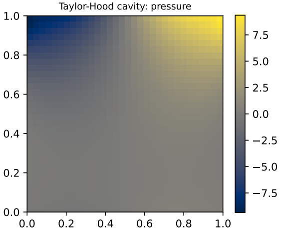
| $\nu$ | $1$ | $10^{-2}$ | $10^{-4}$ |
|---|---|---|---|
| iterations | $27$ | $31$ | $35$ |

The preconditioned spectrum does not move: Theorem 45 is $\nu$-free as stated. The mild growth is a property of the data, not of the preconditioner — the manufactured load $-\nu\Delta\boldsymbol{u}^* + \nabla p^*$ changes direction with $\nu$; with the load held fixed across $\nu$ the counts do not grow. This robustness concerns constant $\nu$; variable viscosity and other parameter regimes require the references of frame 29.

# B6 — $Q_1$–$P_0$ beside Hood–Taylor




$Q_1$–$P_0$, $\varepsilon$-shift: amplitude $\approx 763$




Hood–Taylor: pressure range

$$[-9.86, 9.86]$$

Identical cavity, identical resolution, identical plotting pipeline; the rerun is live (add-back 6). The Lecture-1 history footnote applies: the checkerboard was filtered in practice long before it was understood in theory.