ON THE PROBLEM OF SPURIOUS EIGENVALUES IN THE APPROXIMATION OF LINEAR ELLIPTIC PROBLEMS IN MIXED FORM

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Abstract. In the approximation of linear elliptic operators in mixed form, it is well-known that the so-called *inf-sup* and *ellipticity in the kernel* property are sufficient (and, in a sense to be made precise, necessary) in order to have good approximation properties and optimal error bounds. One might think, in the spirit of [15] and in consideration of the good behavior of commonly used mixed elements (like Raviart–Thomas or Brezzi–Douglas–Marini elements), that these conditions are also sufficient to ensure good convergence properties for eigenvalues. In this paper we show that this is not the case. In particular we present examples of mixed finite element approximations that satisfy the above properties but exhibit spurious eigenvalues. Such bad behavior is proved analytically and demonstrated in numerical experiments. We also present additional assumptions (fulfilled by the already quoted commonly used mixed methods) which guarantee optimal error bounds for eigenvalue approximations as well.

1. Introduction

We consider, as a model problem, the eigenvalue problem for Laplace operator

$$-\Delta u = \lambda u \tag{1.1}$$

in a convex polygonal domain Ω with suitable boundary conditions (to fix ideas, zero Dirichlet boundary conditions). Here and in the following we will always implicitly assume that eigenvectors (here u) are looked for among nonzero functions or vectors. We are interested in the approximation of eigenvalue/eigenvector pairs in the so called *mixed formulation* that reads:

$$\begin{aligned}&\text{find } (\underline{\sigma}, u, \lambda) \text{ in } H(\mathrm{div};\Omega)\times L^2(\Omega)\times\mathbb{R} \text{ such that}\\ &\begin{cases}(\underline{\sigma},\underline{\tau}) + b(\underline{\tau},u) = 0 & \forall \underline{\tau}\in H(\mathrm{div};\Omega)\\ b(\underline{\sigma},v) = -\lambda(u,v) & \forall v\in L^2(\Omega),\end{cases}\end{aligned} \tag{1.2}$$

where, as usual, $(\cdot,\cdot)$ is the inner product in $L^2(\Omega)$ or in $L^2(\Omega)^2$ and $b(\underline{\tau},v) = (\mathrm{div}\,\underline{\tau},v)$. Given finite dimensional subspaces Σ_h and V_h of $H(\mathrm{div};\Omega)$ and $L^2(\Omega)$ respectively, we consider the approximate problem

$$\begin{aligned}&\text{find } (\underline{\sigma}_h, u_h, \lambda_h) \text{ in } \Sigma_h\times V_h\times\mathbb{R} \text{ such that}\\ &\begin{cases}(\underline{\sigma}_h,\underline{\tau}_h) + b(\underline{\tau}_h,u_h) = 0 & \forall \underline{\tau}_h\in \Sigma_h\\ b(\underline{\sigma}_h,v_h) = -\lambda_h(u_h,v_h) & \forall v_h\in V_h.\end{cases}\end{aligned} \tag{1.3}$$

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We point out explicitly that the study of the properties of the mixed eigenvalue problem (1.3) enters as a crucial ingredient in the analysis of more complicated applied problems, such as fluid-structure interactions (see e.g. [2, 18, 11]) or wave-guide resonance (see e.g. [3, 19, 5, 6]) where, in general, one cannot approximate the problem in the easier and more conventional form (1.1).

We assume that the choice of Σ_h and V_h satisfies the usual stability conditions for mixed discretizations. These are the *inf-sup* condition:

$$\text{there exists } \beta > 0, \text{ independent of } h, \text{ such that} \\ \inf_{v_h \in V_h} \sup_{\underline{\tau}_h \in \Sigma_h} \frac{b(\underline{\tau}_h, v_h)}{||\underline{\tau}_h||_{H(\mathrm{div};\Omega)} ||v_h||_{L^2(\Omega)}} \geq \beta \tag{1.4}$$

and the *ellipticity in the Kernel* that is:

$$\text{there exists } \alpha > 0 \text{ independent of } h, \text{ such that} \\ (\underline{\tau}, \underline{\tau}) \geq \alpha ||\underline{\tau}||^2_{H(\mathrm{div};\Omega)} \quad \forall \underline{\tau} \in I\!K_h \tag{1.5}$$

where the discrete kernel $I\!K_h$ is defined as:

$$I\!K_h = \{\underline{\tau} \in \Sigma_h \text{ such that } b(\underline{\tau}, v) = 0 \ \forall v \in V_h\}.$$

One might think (in the spirit of [15]: (3.12–16) and Section 7.a) that the above conditions are sufficient in order to give good approximation properties for eigenvalue/eigenvector pairs, whenever Σ_h and V_h are reasonably good approximations of $H(\mathrm{div};\Omega)$ and $L^2(\Omega)$ respectively. However, this is not the case, as we are going to show in this paper. The reason of failure is hidden in the definition of the compact operator whose spectrum has to be approximated (here the inverse of Laplace operator), when the mixed formulation is used. To make things clearer, let us introduce the associated boundary value problem in its usual form and in the mixed formulation. Let therefore f be given in $L^2(\Omega)$, and consider the problem:

$$\text{find } u \in H^1_0(\Omega) \text{ such that} \\ -\Delta u = f \quad \text{in } \Omega. \tag{1.6}$$

The unique solution of this problem defines a linear compact operator T from $L^2(\Omega)$ into itself: $u = Tf$. Consider the same problem in its mixed formulation: now, for a given $f \in L^2(\Omega)$, we are looking for a pair $(\underline{\sigma}, u)$ in $H(\mathrm{div};\Omega) \times L^2(\Omega)$ that satisfies

$$\begin{cases} (\underline{\sigma}, \underline{\tau}) + b(\underline{\tau}, u) = 0 & \forall \underline{\tau} \in H(\mathrm{div};\Omega) \\ b(\underline{\sigma}, v) = -(f, v) & \forall v \in L^2(\Omega). \end{cases} \tag{1.7}$$

Clearly, the u part of the mixed formulation is still given by $u = Tf$, while $\underline{\sigma}$ is just the gradient of u. However, to be precise, we have now *another* operator (say T_M) which is acting from $L^2(\Omega)$ into $H(\mathrm{div};\Omega) \times L^2(\Omega)$. This is not a good setting if we want to look for eigenvalues. Therefore [15], following [14], considers first the product space $H = H(\mathrm{div};\Omega) \times L^2(\Omega)$ and the operator T_H, from H' into H defined as follows:

$$\text{given } (g, f) \text{ in } H', \text{ find } (\underline{\sigma}, u) \text{ in } H \text{ such that} \\ \begin{cases} (\underline{\sigma}, \underline{\tau}) + b(\underline{\tau}, u) = < g, \underline{\tau} > & \forall \underline{\tau} \in H(\mathrm{div};\Omega) \\ b(\underline{\sigma}, v) = -(f, v) & \forall v \in L^2(\Omega). \end{cases}$$

Then they consider the cutoff operator (say, F) from $L^2(\Omega)^2 \times L^2(\Omega)$ into itself, given by $F(g, f) = (0, f)$, and they are lead to a generalized eigenvalue problem of the type

$$(\underline{\sigma}, u) = \lambda T_H F(\underline{\sigma}, u).$$

This is surely correct, but now the operator $T_H F$, from H into itself, is not compact anymore and all the subsequent theory of [15] does not apply. We point out that the reason of failure is not originated by an inconvenient way of writing the eigenvalue problem: indeed, the operator T_M itself, (mapping f into $(\underline{\sigma}, u)$) is not compact as an operator from $L^2(\Omega)$ in H.

However the results of [15] (Section 7.a) are true (see for instance [1]), since additional properties (besides the inf-sup and the ellipticity in the Kernel) hold for their choice of finite element spaces Σ_h and V_h, which make the method work. On the other hand other reasonable choices of Σ_h and V_h, although verifying the *inf-sup* (1.4) and the *ellipticity in the Kernel* (1.5) properties, fail miserably when applied to eigenvalue problems as we shall prove analytically and demonstrate by numerical experiments.

An outline of the paper is the following. In Section 2 we present an abstract framework in which the problem can be set and we define in a precise way what is to be considered as good convergence property for eigenvalue/eigenvector pairs.

In Section 3 we recall four types of choices for the spaces Σ_h and V_h: the *truly mixed* approach, the $Q_1 - P_0$ element on rectangular grids, the $P_1 - \operatorname{div}(P_1)$ element and the $P_1^* - Q_0$ element on *criss-cross* grids. We show in particular that the last two elements satisfy both the *inf-sup* and the *ellipticity in the Kernel* properties. The same is already well-known for the *truly mixed* approach, while the $Q_1 - P_0$ element is only used as an auxiliary step for studying the others, although, being a well-known element, it deserves an analysis for itself: in particular we show that this element, which does not verify the usual *inf-sup* condition for Stokes problem, does indeed verify a sort of *inf-sup* condition in $H(\operatorname{div};\Omega)$ that might be of some interest in other applications.

Numerical experiments, reported in Section 4, show however that only the *truly mixed* approach gives good discrete eigenvalues, while the others exhibit the presence of spurious ones. We stress the fact that the type of failure exhibited by approximations like $P_1^* - Q_0$ or $P_1 - \operatorname{div}(P_1)$ is, in practice, much more dangerous than the type of failure normally exhibited by choices that do not satisfy the *inf-sup* condition. Indeed, the latter elements usually have a cloud of spurious eigenvalues that immediately shows the bad quality of the computation. On the other hand, as it will appear clearer from the numerical experiments shown in Section 4, the former elements have just a few, well-isolated spurious eigenvalues that, when we look at the discrete spectrum, insidiously look as good ones. In Section 5 we prove analytically that the above elements (with the obvious exception of the *truly mixed* ones), must fail when used to approximate eigenvalues, thus confirming the numerical results of Section 4. Finally, in Section 6, we give some simple sufficient conditions for having good convergence properties of eigenvalue/eigenvector pairs. These sufficient conditions include the usual *inf-sup* condition and the *ellipticity in the Kernel* plus an additional property regarding the so called *Fortin operator* (see [10]). The *truly mixed* approach satisfies this last property, so that the theory confirms again the numerical results of Section 4. More general sufficient conditions and additional references can be found in [16] and [1].

2. Setting of the problem

We are interested in the approximation of the following eigenproblem:

$$
\text{find } (\underline{\sigma}, u, \lambda) \in H(\mathrm{div};\Omega) \times L^2(\Omega) \times \mathbb{R} \text{ such that}
\tag{2.1}
$$
$$
\begin{cases} (\underline{\sigma}, \underline{\tau}) + (\mathrm{div}\, \underline{\tau}, u) = 0 & \forall \underline{\tau} \in H(\mathrm{div};\Omega) \\ (\mathrm{div}\, \underline{\sigma}, v) = -\lambda(u, v) & \forall v \in L^2(\Omega). \end{cases}
$$

Given finite dimensional subspaces $\Sigma_h \subset H(\mathrm{div};\Omega)$ and $V_h \subset L^2(\Omega)$, the discretization of (2.1) is

$$
\text{find } (\underline{\sigma}_h, u_h, \lambda_h) \in \Sigma_h \times V_h \times \mathbb{R} \text{ such that}
\tag{2.2}
$$
$$
\begin{cases} (\underline{\sigma}_h, \underline{\tau}_h) + (\mathrm{div}\, \underline{\tau}_h, u_h) = 0 & \forall \underline{\tau}_h \in \Sigma_h \\ (\mathrm{div}\, \underline{\sigma}_h, v_h) = -\lambda_h(u_h, v_h) & \forall v_h \in V_h. \end{cases}
$$

Let $T : L^2(\Omega) \to L^2(\Omega)$ be the self-adjoint compact operator defined by

$$
\begin{cases} (\underline{\sigma}, \underline{\tau}) + (\mathrm{div}\, \underline{\tau}, Tf) = 0 & \forall \underline{\tau} \in H(\mathrm{div};\Omega) \\ (\mathrm{div}\, \underline{\sigma}, v) = -(f, v) & \forall v \in L^2(\Omega). \end{cases}
\tag{2.3}
$$

Then $(\underline{\sigma}, u, \lambda)$ is an eigensolution of problem (2.1) if and only if

$$
\lambda T u = u, \quad \underline{\sigma} = \nabla u.
\tag{2.4}
$$

Hence the eigenvalues λ_i $(i \in \mathbb{N})$ of problem (2.1) are positive. We denote them by

$$
\begin{aligned} &0 < \lambda_1 \le \lambda_2 \le \dots \le \lambda_i \le \dots \\ &\lim_{i\to\infty} \lambda_i = +\infty \end{aligned}
\tag{2.5}
$$

For each $i \in \mathbb{N}$ the algebraic multiplicity of λ_i is one, and E_i is the one-dimensional eigenspace associated to λ_i. In $L^2(\Omega)$ we introduce an orthonormal basis $\{u_i\}$ such that

$$
\begin{aligned} &E_i = \mathrm{span}(u_i) \\ &(u_i, u_j) = \delta_{ij}. \end{aligned}
\tag{2.6}
$$

The following mapping will be useful in the following. Let $m : \mathbb{N} \to \mathbb{N}$ be the application which to every N associates the dimension of the space generated by the eigenspaces of the first N distinct eigenvalues; that is

$$
\begin{aligned} &m(1) = \dim\{\oplus_i E_i : \lambda_i = \lambda_1\}, \\ &m(N+1) = m(N) + \dim\{\oplus_i E_i : \lambda_i = \lambda_{m(N)+1}\}. \end{aligned}
\tag{2.7}
$$

Clearly, $\lambda_{m(1)}, \dots, \lambda_{m(N)}$ $(N \in \mathbb{N})$ will now be the first N *distinct* eigenvalues of (2.1).

Let us denote by $T_h : L^2(\Omega) \to L^2(\Omega)$ the discrete counterpart of T, defined by

$$
\begin{cases} (\underline{\sigma}_h, \underline{\tau}_h) + (\mathrm{div}\, \underline{\tau}_h, T_h f) = 0 & \forall \underline{\tau}_h \in \Sigma_h \\ (\mathrm{div}\, \underline{\sigma}_h, v_h) = -(f, v_h) & \forall v_h \in V_h. \end{cases}
\tag{2.8}
$$

Then $\{T_h\}$ is a family of self-adjoint compact operators with finite-dimensional range in $L^2(\Omega)$. As in the continuous case $(\underline{\sigma}_h, u_h, \lambda_h) \in \Sigma_h \times V_h \times \mathbb{R}$ is an eigensolution of problem (2.2) if and only if

$$
\lambda_h T_h u_h = u_h, \quad \underline{\sigma}_h = \nabla_h u_h,
\tag{2.9}
$$

with ∇_h a suitable discretization of ∇.

Let $\dim V_h = N(h)$; then T_h admits $N(h)$ real positive eigenvalues

$$
\lambda_1^h \le \dots \le \lambda_i^h \le \dots \lambda_{N(h)}^h.
\tag{2.10}
$$

The associated discrete eigenfuntions u_i^h, $i = 1, \dots, N(h)$ give rise to an orthonormal basis in V_h with respect to the scalar product of $L^2(\Omega)$. Let $E_i^h = \mathrm{span}(u_i^h)$ denote the discrete eigenspace associated to λ_i^h.

A classical assumption in the theory of spectrum perturbation is the uniform convergence of the operators, that is

$$\lim_{h\to 0} ||T - T_h||_{\mathcal{L}(L^2(\Omega))} = 0. \tag{2.11}$$

As a consequence of (2.11), we have

$$\begin{aligned} &\forall\varepsilon > 0,\ \forall N \in \mathbb{N} \quad \exists h_0 > 0 \text{ such that } \forall h \le h_0 \\ &\max_{i=1,\dots,m(N)} |\lambda_i - \lambda_i^h| \le \varepsilon, \\ &\hat{\delta}(\oplus_{i=1}^{m(N)} E_i, \oplus_{i=1}^{m(N)} E_i^h) \le \varepsilon, \end{aligned} \tag{2.12}$$

where $\hat{\delta}(E, F)$, for E and F linear subspaces of $L^2(\Omega)$, represents the gap between E and F and is defined by

$$\begin{aligned} \hat{\delta}(E,F) &= \max[\delta(E,F), \delta(F,E)], \\ \delta(E,F) &= \sup_{u\in E,\ ||u||=1} \inf_{v\in F} ||u - v||_0, \end{aligned} \tag{2.13}$$

In (2.13) $||\cdot||_0$ stands for the L^2-norm.

We conclude with an additional notation that will be constantly used in the following. Although the definition of the space $V_h \subset L^2(\Omega)$ will change from one example of finite element approximation to the others, we shall always denote by the symbol $I\!P_h$ the $L^2(\Omega)$–projection onto V_h, that is

$$\int_\Omega (v - I\!P_h v) v_h \, dx = 0 \quad \forall v \in L^2(\Omega),\ \forall v_h \in V_h. \tag{2.14}$$

3. Various choices of spaces

In this section we present several possible choices for the spaces $\Sigma_h \subset H(\mathrm{div};\Omega)$ and $V_h \subset L^2(\Omega)$. For each choice of spaces, we test the validity of the following two hypotheses:

$$\begin{aligned} &\text{there exists } \alpha > 0 \text{ independent of } h, \text{ such that} \\ &(\underline{\tau}, \underline{\tau}) \ge \alpha ||\underline{\tau}||^2_{\mathrm{div}} \quad \forall \underline{\tau} \in I\!K_h, \end{aligned} \tag{3.1}$$

where the discrete kernel $I\!K_h$ is defined as

$$I\!K_h = \{\underline{\tau} \in \Sigma_h \text{ such that } b(\underline{\tau}, v) = 0\ \forall v \in V_h\}$$

and

$$\begin{aligned} &\text{there exists } \beta > 0, \text{ independent of } h, \text{ such that} \\ &\inf_{v_h\in V_h} \sup_{\underline{\tau}_h\in\Sigma_h} \frac{b(\underline{\tau}_h, v_h)}{||\underline{\tau}_h||_{\mathrm{div}} ||v_h||_0} \ge \beta. \end{aligned} \tag{3.2}$$

In (3.1) and (3.2) $||\cdot||_{\mathrm{div}}$ denotes the graph norm in $H(\mathrm{div};\Omega)$.

It is well-known that the assumptions (3.1) and (3.2) ensure the existence, uniqueness and stability of the solution of (2.8) (see [8]). We shall see that these hypotheses are not sufficient in order to have a good mixed approximation of the spectrum for Laplace operator.

3.1. **The mixed approach.** Let us consider classical approximations of $H(\mathrm{div};\Omega)$, among which there are, for instance, the elements introduced by Raviart–Thomas (RT), Brezzi–Douglas–Marini (BDM) and Brezzi–Douglas–Fortin–Marini (BDFM). For a unified presentation we refer to [8]. In this subsection Σ_h will be one of the mixed finite element spaces quoted above. Correspondingly V_h will be the space $\mathrm{div}\,\Sigma_h$, which contains piecewise polynomials of a certain degree k.

We recall the main properties which are enjoied by the pair (Σ_h, V_h) and which turn out to be crucial for the eigenvalue approximation.

The first property concerns the so-called *Fortin's operator* Π_h, acting from $W := H(\mathrm{div};\Omega) \cap (L^s(\Omega))^2$ ($s > 2$ fixed) into Σ_h. Such operator, defined using suitable degrees of freedom, verifies the *commuting diagram*:

$$
\begin{tikzcd}
W \arrow[r, "\mathrm{div}"] \arrow[d, "\Pi_h"'] & L^2(\Omega) \arrow[d, "\mathbb{P}_h"] \\
\Sigma_h \arrow[r, "\mathrm{div}"'] & V_h
\end{tikzcd}
\tag{3.3}
$$

This implies that (3.1) and (3.2) are satisfied (see e.g. [8], pag. 131).

The following approximation property holds for $1 \le m \le k+1$:

$$
||\underline{\tau} - \Pi_h \underline{\tau}||_0 \le C h^m |\underline{\tau}|_m. \tag{3.4}
$$

Let $f \in L^2(\Omega)$, then, due to the regularity assumptions on Ω, Tf belongs to $H^2(\Omega)$. Hence we have (see [8], (IV.1.31))

$$
||Tf - T_h f||_0 \le Ch(||\underline{\nabla} Tf||_1 + ||Tf||_1). \tag{3.5}
$$

This last equation means that T_h converges uniformly to T, see (2.11).

3.2. **The $Q_1 - P_0$ element on a rectangular mesh.** Let us consider a square domain Ω and a partition of Ω into $N \times N$ macroelements, each made of 2×2 squares. As usual K will denote an element (of length h) of the triangulation $\mathcal{T}_h$.

We consider the following approximating spaces:

$$
\begin{aligned}
\Sigma_Q &= \{\underline{\tau}_h \in [C^0(\Omega)]^2 : \underline{\tau}_h|_K \in [Q_1(K)]^2 \ \forall K \in \mathcal{T}_h\}, \\
V_h &= \{v_h : v_h|_K \in P_0(K) \ \forall K \in \mathcal{T}_h\}.
\end{aligned} \tag{3.6}
$$

This choice of spaces does not verify the inf-sup condition (3.2). However, we prove a modified inf-sup condition involving a mesh dependent norm. This result will be useful in order to analyze the element of the next subsection.

A local basis of V_h on a macroelement is shown in Fig. 1. Notice that the basis we have chosen is orthogonal.

Let V_{JP} be the subspace of V_h locally generated by v_i, $i = 1, 2, 3$. In the paper by Johnson and Pitkäranta [13], it has been proved that the spaces Σ_Q and V_{JP} verify the inf-sup condition as follows.

Lemma 3.1. *There exists a constant C independent of h such that*

$$
\sup_{\underline{\tau}_h \in \Sigma_Q} \frac{(\mathrm{div}\,\underline{\tau}_h, v_h)}{||\underline{\tau}_h||_1} \ge C||v_h||_0 \tag{3.7}
$$

for all $v_h \in V_{JP}$.

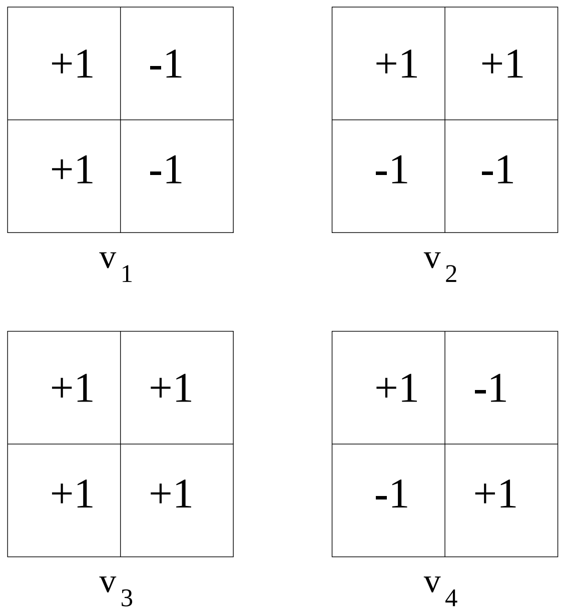

FIGURE 1. Basis for V_h on a macroelement of 2×2 squares

In Lemma 3.1 the space V_{JP} cannot be replaced by the space V_h. However, if the norm in $H^1(\Omega)$ is substituted by a mesh dependent one, then it is possible to verify the inf-sup condition. We set

$$||\underline{\tau}_h||_h = (||\underline{\tau}_h||_0^2 + ||\mathbb{P}_h \operatorname{div} \underline{\tau}_h||_0^2)^{1/2}; \tag{3.8}$$

then the following theorem holds true.

Theorem 3.2. *There exists a constant C independent of h such that for every $v_h \in V_h$ there exists $\underline{\tau}_h \in \Sigma_Q$ verifying*

$$\begin{aligned} &(\operatorname{div} \underline{\tau}_h, v_h) \geq ||v_h||_0^2 \\ &||\underline{\tau}_h||_h \leq C||v_h||_0. \end{aligned} \tag{3.9}$$

Proof. We work on macroelements of 2×2 squares. Let us split a given $v_h \in V_h$ into the sum of $v_h^b \in V_{JP}$ and v_h^c which is locally generated by v_4 (see Fig. 1), so that $v_h = v_h^c + v_h^b$.

Using Lemma 3.1, there exists $\underline{\tau}_h^b \in \Sigma_Q$ such that

$$\begin{aligned} &(\operatorname{div} \underline{\tau}_h^b, v_h^b) \geq ||v_h^b||_0^2, \\ &||\underline{\tau}_h^b||_1 \leq C_1||v_h^b||_0. \end{aligned} \tag{3.10}$$

The main step of the proof is to construct an element $\underline{\tau}_h^c \in \Sigma_Q$ such that

$$\begin{aligned} &\mathbb{P}_h \operatorname{div} \underline{\tau}_h^c = v_h^c \\ &||\underline{\tau}_h^c||_0 \leq C_2||v_h^c||_0. \end{aligned} \tag{3.11}$$

We fix our attention to the row of macroelements lying in the strip $S_j =]0, 2Nh[\times]2(j-1)h, 2jh[$. On each macroelement, v_h^c is equal to v_4 multiplied by a certain constant. We denote by c_i the value of such constant on the $i-th$ macroelement, $i = 1, \dots, N$. In the row we have considered, we define $\underline{\tau}_h^c$ using

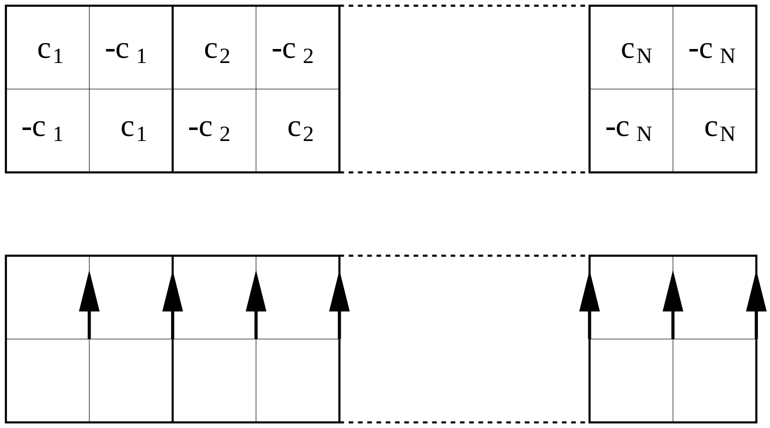

FIGURE 2. Degrees of freedom for a row of checkerboards

the $2N$ degrees of freedom drawn in Fig. 2. In all other nodes it is set equal to zero. Since v_h^c is piecewise constant, an explicit computation shows that $\underline{\tau}_h^c$ can be defined as follows, in order to have $(\mathbb{P}_h \operatorname{div} \underline{\tau}_h^c)|_K = (v_h^c)|_K$ for each square $K \subset S_j$:

$$\underline{\tau}_h^c((2i-1)h,(2j-1)h) = 2h(0, -c_i - 2\sum_{\ell=1}^{i-1} c_\ell)$$

$$\underline{\tau}_h^c(2ih,(2j-1)h) = 2h(0, 2\sum_{\ell=1}^{i} c_\ell).$$

The $L^2(\Omega)$-norm in S_j of v_h^c and $\underline{\tau}_h^c$ can be evaluated as follows:

$$\begin{aligned}
||v_h^c||_{0,S_j}^2 &= 4h^2 \sum_{i=1}^{N} c_i^2 \\
||\underline{\tau}_h^c||_{0,S_j}^2 &\le Ch^4 \sum_{i=1}^{N} \left(\sum_{\ell=1}^{i} c_\ell \right)^2 \\
&\le Ch^4 \sum_{i=1}^{N} \left(\sum_{\ell=1}^{i} 1 \right) \left(\sum_{\ell=1}^{i} c_\ell^2 \right) \le Ch^2 \sum_{i=1}^{N} c_i^2.
\end{aligned}$$

Then we have

$$||\underline{\tau}_h^c||_{0,S_j}^2 \le C||v_h^c||_{0,S_j}^2, \tag{3.12}$$

which implies the corresponding bound in the whole domain (3.11).

We are now in position to conclude the proof. Let $\gamma = (1 + C_1^2)/2$, where C_1 is defined in (3.10). Taking $\underline{\tau}_h = \gamma \underline{\tau}_h^c + \underline{\tau}_h^b$ and noting that $(\operatorname{div} \underline{\tau}_h^c, v_h^b) = (v_h^c, v_h^b) = 0$,

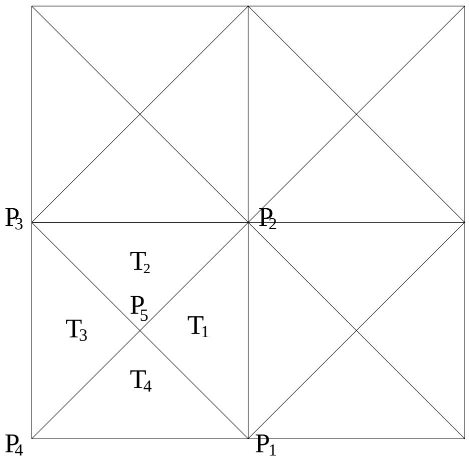

FIGURE 3. A criss-cross macroelement

we obtain

$$
\begin{aligned}
(\operatorname{div} \underline{\tau}_h, v_h) &= \gamma(\operatorname{div} \underline{\tau}_h^c, v_h^c) + (\operatorname{div} \underline{\tau}_h^b, v_h^b) + (\operatorname{div} \underline{\tau}_h^b, v_h^c) \\
&\geq \gamma ||v_h^c||_0^2 + ||v_h^b||_0^2 - ||\operatorname{div} \underline{\tau}_h^b||_0 ||v_h^c||_0 \geq \gamma ||v_h^c||_0^2 + ||v_h^b||_0^2 - C_1 ||v_h^b||_0 ||v_h^c||_0 \\
&\geq \left(\gamma - \frac{C_1^2}{2}\right) ||v_h^c||_0^2 + \frac{1}{2} ||v_h^b||_0^2 \geq \frac{1}{2} ||v_h||_0^2, \\
||\underline{\tau}_h||_h^2 &\leq \gamma ||\underline{\tau}_h^c||_0^2 + \gamma ||I\!P_h \operatorname{div} \underline{\tau}_h^c||_0^2 + ||\underline{\tau}_h^b||_1^2 \leq C(||v_h^c||_0^2 + ||v_h^b||_0^2) = C||v_h||_0^2.
\end{aligned}
$$

□

Remark 3.3. The result of Theorem 3.2 does not imply the inf-sup condition (3.2), since in (3.9) the mesh dependent norm $||\cdot||_h$ is used instead of the norm of $H(\mathrm{div};\Omega)$. Theorem 3.2 is however crucial for the analysis of the element presented in the next subsection.

3.3. **The $P_1 - \mathrm{div}(P_1)$ element on a criss-cross mesh.** It is well-known that the $P_1 - P_0$ element does not satisfy the inf-sup condition (3.2) on a criss-cross mesh (see, for instance, [8]). Indeed there exists a piecewise constant function which is orthogonal to the divergence of every continuous piecewise linear vector fields. Hence we define V_h to be the space of the divergences of all continuous piecewise linear vector fields.

For this element we are able to prove both the *ellipticity in the Kernel* (3.1) and the *inf-sup* (3.2) conditions.

Let us consider a square domain Ω, which is split into $2N \times 2N$ squares; each of them is then partitioned into four triangles by its diagonals (see Fig. 3). We denote by $Q \in \mathcal{Q}_h$ the squares and by $T \in \mathcal{T}_h$ the triangles.

We introduce the following finite element spaces:

$$
\begin{aligned}
&\Sigma_h = \{\underline{\tau}_h \in [C^0(\Omega)]^2 : \underline{\tau}_h|_T \in [P_1(T)]^2 \ \forall T \in \mathcal{T}_h\}, \\
&V_h = \operatorname{div}(\Sigma_h).
\end{aligned}
\tag{3.13}
$$

In the following theorem we observe that our choice (3.13) satisfies (3.1).

Theorem 3.4. *The spaces Σ_h and V_h defined in (3.13) satisfy the* ellipticity in the Kernel *property (3.1).*

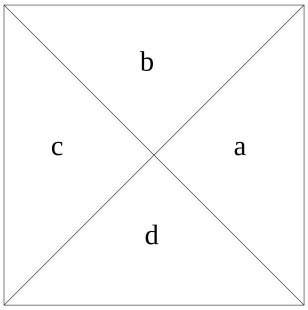

FIGURE 4. Piecewise constants on a criss-cross square

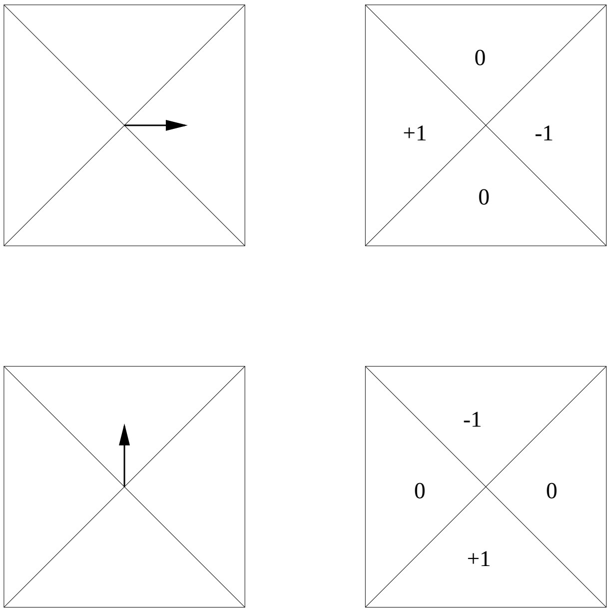

FIGURE 5. The divergence of B

Proof. The discrete kernel $I\!K_h$, due to the definition of V_h, is contained in the continuous kernel. □

In the following lemma, we characterize the space V_h.

Lemma 3.5. *The elements of V_h are piecewise constants and are characterized by the following relation between the values on each triangle in a criss-cross square (see Fig. 4):*

$$a + c = b + d. \tag{3.14}$$

Proof. In Figs. 5 and 6 the divergence of some basis function in Σ_h are represented. By linearity the result follows immediately for the divergence of every vector in Σ_h. □

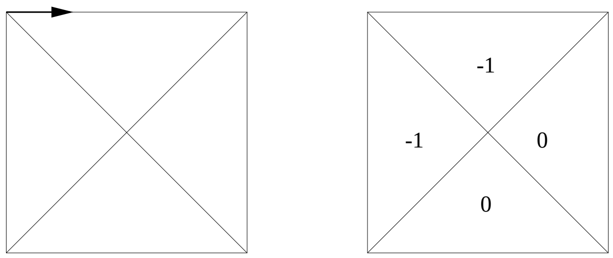

FIGURE 6. The divergence of an element in Σ_h

We set

$$V_c = \{v \in V_h : v|_Q \text{ is constant } \forall Q \in \mathcal{Q}_h\} \tag{3.15}$$

$$V_b = \{v \in V_h : \int_Q v = 0 \ \forall Q \in \mathcal{Q}_h\}. \tag{3.16}$$

It is immediate to see that V_b, in each square Q, has dimension two, a basis being given by the two modes on the right-hand side of Fig. 5.

Lemma 3.6. *The following orthogonal decomposition holds true:*

$$V_h = V_c \oplus V_b, \quad \text{with } V_c \perp V_b. \tag{3.17}$$

Proof. Given $v \in V_h$, let us consider the element $v_c \in V_c$ such that

$$\int_Q v_c = \int_Q v \quad \forall Q \in \mathcal{Q}_h. \tag{3.18}$$

Then $v_b = v - v_c$ is an element of V_b. It is obvious that with this construction the decomposition is unique. Moreover,

$$\int_\Omega v_c v_b = 0. \tag{3.19}$$

□

We set

$$B = \operatorname{span}\{\underline{\tau} \in \Sigma_h : \operatorname{supp} \underline{\tau} \subseteq Q,\ Q \in \mathcal{Q}_h\}. \tag{3.20}$$

The divergence of a local basis in B is represented in Fig. 5. The inclusion $\operatorname{div} B \subseteq V_b$ is obvious.

The following Lemma is also obvious from Fig. 5 and a simple scaling argument.

Lemma 3.7. *The divergence operator is injective and surjective between B and V_b. That is for each $v_b \in V_b$ there exists a unique $\underline{b} \in B$ which satisfies*

$$\operatorname{div} \underline{b} = v_b. \tag{3.21}$$

Moreover there exists C independent of h such that

$$||\underline{b}||_0 \le Ch||v_b||_0. \tag{3.22}$$

We set

$$\Sigma_c = \{\underline{\tau} \in \Sigma_h : \operatorname{div} \underline{\tau} \in V_c\} \tag{3.23}$$

Lemma 3.8. *The following decomposition holds true:*

$$\Sigma_h = \Sigma_c \oplus B. \tag{3.24}$$

Moreover the following orthogonalities are verified:

$$\begin{aligned} (\operatorname{div} \underline{\tau}_c, v_b) = 0 \quad & \forall \underline{\tau}_c \in \Sigma_c,\ \forall v_b \in V_b, \\ (\operatorname{div} \underline{b}, v_c) = 0 \quad & \forall \underline{b} \in B,\ \forall v_c \in V_c. \end{aligned} \tag{3.25}$$

Proof. Let us consider $\underline{\tau} \in \Sigma_h$. By definition of V_h and Lemma 3.6 there exist $v_c \in V_c$ and $v_b \in V_b$ such that $\operatorname{div} \underline{\tau} = v_c + v_b$. Let $\underline{b}$ be the unique element of B such that $\operatorname{div} \underline{b} = v_b$ (see Lemma 3.7). We define $\underline{\tau}_c = \underline{\tau} - \underline{b}$. It results that $\operatorname{div} \underline{\tau}_c = v_c$, hence $\underline{\tau}_c \in \Sigma_c$. The decomposition is unique by construction. Finally, the orthogonalities (3.25) are straightforward to verify. □

The following lemma will be useful in order to apply the results of the previous subsection.

Lemma 3.9. *Let Σ_Q be the space defined in (3.6) (that is, locally Q_1). Then for each $\underline{\xi}_h \in \Sigma_Q$ there exists $\underline{\tau}_h \in \Sigma_c$ satisfying*

$$(\operatorname{div} \underline{\tau}_h, v) = (\operatorname{div} \underline{\xi}_h, v) \quad \forall v \in V_c, \tag{3.26}$$

$$||\underline{\tau}_h||_r \le C ||\underline{\xi}_h||_r, \quad (r = 0, 1), \tag{3.27}$$

with C independent of h. Moreover $\underline{\tau}_h$ can be chosen such that it attains the same values as $\underline{\xi}_h$ at all nodes of $\mathcal{Q}_h$.

Proof. For a $\underline{\tau}_h \in \Sigma_c$ equation (3.26) means that $\operatorname{div} \underline{\tau}_h$ is the $L^2(\Omega)$–projection of $\operatorname{div} \underline{\xi}_h$ onto V_c. Let Q be a square in $\mathcal{Q}_h$ (see Fig. 3). Let us denote by φ_i, $i = 1, \dots, 4$, the piecewise linear basis functions associated to the vertices P_1, P_2, P_3 and P_4 of Q and by φ_5 the one associated to the center. On Q, $\underline{\tau}_h \in \Sigma_h$ can be represented as follows:

$$\underline{\tau}_h = \sum_{i=1}^{5} (u_i, v_i) \varphi_i. \tag{3.28}$$

We take

$$(u_i, v_i) = \underline{\xi}_h(P_i), \quad i = 1, \dots, 4. \tag{3.29}$$

Whatever the value of (u_5, v_5) may be, the mean-value of $\operatorname{div} \underline{\tau}_h$ on Q is equal to the mean-value of $\operatorname{div} \underline{\xi}_h$ thanks to the Gauss theorem. Hence condition (3.26) is satisfied. We have only to fix the value of $\underline{\tau}_h$ in P_5 in order to achieve that it belongs to Σ_c. A straightforward calculation leads to

$$\begin{aligned} u_5 &= (u_1 - v_1 + u_2 + v_2 + u_3 - v_3 + u_4 + v_4)/4, \\ v_5 &= (-u_1 + v_1 + u_2 + v_2 - u_3 + v_3 + u_4 + v_4)/4. \end{aligned} \tag{3.30}$$

A scaling argument gives the bounds (3.27). □

We state the main result of this section.

Theorem 3.10. *The following inf-sup condition holds true:*

$$\inf_{v \in V_h} \sup_{\underline{\tau} \in \Sigma_h} \frac{(\operatorname{div} \underline{\tau}, v)}{||\underline{\tau}||_{\operatorname{div}} ||v||_0} \ge \beta_0 > 0, \quad \forall h > 0. \tag{3.31}$$

Proof. Given $v_h \in V_h$, let $v_c \in V_c$ and $v_b \in V_b$ be such that $v_h = v_c + v_b$. Theorem 3.2 and Lemma 3.7 imply the existence of $\underline{\xi}_h \in \Sigma_Q$ and $\underline{b} \in B$, respectively, satisfying

$$\begin{array}{ll}(\operatorname{div} \underline{\xi}_h, v_c) \geq ||v_c||_0^2 & (\operatorname{div} \underline{b}, v_b) = ||v_b||_0^2 \\ ||\underline{\xi}_h||_{\operatorname{div}} \leq C_1 ||v_c||_0 & ||\underline{b}||_{\operatorname{div}} \leq C_2 h ||v_b||_0.\end{array} \tag{3.32}$$

Using Lemma 3.9 there exists $\underline{\tau}_c \in \Sigma_c$ such that

$$\begin{array}{l}(\operatorname{div} \underline{\tau}_c, v_c) = (\operatorname{div} \underline{\xi}_h, v_c) \\ ||\underline{\tau}_c||_{H(\operatorname{div};\Omega)} \leq C ||\underline{\xi}_h||_{H(\operatorname{div};\Omega)}.\end{array} \tag{3.33}$$

It is not difficult to verify that, defining $\underline{\tau}_h = \underline{\tau}_c + \underline{b}$, thanks to Lemma 3.8, one has:

$$\begin{array}{l}(\operatorname{div} \underline{\tau}_h, v_h) \geq ||v_h||_0^2 \\ ||\underline{\tau}_h||_{\operatorname{div}} \leq C ||v_h||_0.\end{array} \tag{3.34}$$

□

Remark 3.11. Inequality (3.31) is optimal; in fact it cannot be improved, since for each h there exists $\tilde{v}_h \in V_h$ such that

$$(\operatorname{div} \underline{\tau}_h, \tilde{v}_h) \leq C ||\underline{\tau}_h||_0 ||\tilde{v}_h||_0 \quad \forall \underline{\tau}_h \in \Sigma_h. \tag{3.35}$$

This inequality has been proved by Qin in his Ph.D. dissertation [17] using an idea of Boland–Nicolaides [7] (see also [12]). In particular the element $\tilde{v}_h$ is a properly chosen linear combination of checkerboards on the macroelements.

Remark 3.12. In the proof of Theorem 3.2, the normal component of vectors in Σ_h has not been used, while we used the tangential component on the boundary (see Fig. 2). Actually the proof could be completed without using any boundary degrees of freedom for the space Σ_h. It turns out that the spaces $\Sigma_h \cap H_0^1(\Omega)^2$ and $\operatorname{div}(\Sigma_h \cap H_0^1(\Omega)^2)$ verify the inf-sup condition (3.2).

3.4. **The $P_1^* - Q_0$ element on a criss-cross mesh.** Let us consider again the $P_1 - \operatorname{div}(P_1)$ element of the previous section. During the analysis of this element, we introduced the subspace Σ_c (see (3.23) and (3.16)) made, essentially, of P_1 vectors (on a criss-cross grid) where the value at the "cross node" is adjusted in order to have a divergence which is constant in each square. This is what we call P_1^*. Hence we use here $\Sigma_c \subset H(\operatorname{div};\Omega)$ for approximating the vectors and $V_c \subset L^2(\Omega)$ for approximating the scalars (we always refer to definitions (3.23) and (3.16)). Out of Theorem 3.2 and Lemma 3.9 we easily obtain (cfr. also (3.33)) that this choice satisfies the inf-sup condition (3.2). Moreover, as $\operatorname{div}(\Sigma_c) = V_c$, the ellipticity in the kernel property (3.1) will also hold trivially.

4. Numerical results

Let Ω be the square $]0,\pi[\times]0,\pi[$. The following table shows the first frequencies obtained using some of the mixed elements discussed in the previous Section: Raviart–Thomas of lowest degree (RT), $P_1 - \operatorname{div} P_1$ (P1) and $P_1^* - Q_0$ (P1*). For all the elements, the 16×16 criss-cross mesh has been used.

We point out that only RT element gives satisfactory reults. In the other two elements spurious modes appear, which neither converge to any continuous eigenvalue nor tend to zero or to infinity.

We describe this behavior more precisely in Table 2 for the P1 element. We can observe that the fourth numerical eigenvalue seems to converge to 6, which

TABLE 1. Comparison of frequencies for different approximations

mode	exact	RT	P1	P1*
(1,1)	2.00000	1.99786	2.00428	2.01286
(2,1)	5.00000	4.99382	5.02674	5.08056
(1,2)	5.00000	4.99382	5.02674	5.08056
			5.98074	6.03707
(2,2)	8.00000	7.96568	8.06845	8.20593
(3,1)	10.0000	9.99754	10.1067	10.3240
(1,3)	10.0000	9.99754	10.1067	10.3240
(3,2)	13.0000	12.9292	13.1804	13.5448
(2,3)	13.0000	12.9292	13.1804	13.5448
			14.7166	15.0528
			14.7166	15.0528
(4,1)	17.0000	17.0241	17.3073	17.9431
(1,4)	17.0000	17.0241	17.3073	17.9431
(3,3)	18.0000	17.8258	18.3456	19.0411
(4,2)	20.0000	19.8995	20.4254	21.2951
(2,4)	20.0000	19.8995	20.4254	21.2951

TABLE 2. Nodal approximation on criss-cross mesh

exact	computed			
2.00000	2.01711	2.00761	2.00428	2.00274
5.00000	5.10637	5.04748	5.02674	5.01712
5.00000	5.10637	5.04748	5.02674	5.01712
	5.92302	5.96578	5.98074	5.98767
8.00000	8.27150	8.12152	8.06845	8.04383
10.0000	10.4196	10.1890	10.1067	10.0684
10.0000	10.4196	10.1890	10.1067	10.0684
13.0000	13.7043	13.3195	13.1804	13.1156
13.0000	13.7043	13.3195	13.1804	13.1156
	13.9669	14.5093	14.7166	14.8163
	13.9669	14.5093	14.7166	14.8163
17.0000	18.1841	17.5423	17.3073	17.1972
17.0000	18.1841	17.5423	17.3073	17.1972
mesh	8×8	12×12	16×16	20×20

does not belong to the spectrum of the continuous problem. The P1 and P1* elements, even if they verify both conditions (3.1) and (3.2), give poor results for the approximation of problem (2.1). The presence of the spurious eigenvalues can be motivated by the fact that (3.1) and (3.2) are not sufficient conditions to ensure that the eigensolutions are "well approximated". In the next section we state the meaning of "well-approximated eigensolutions" and we give a necessary condition for this property.

Remark 4.1. Although the $Q_1 - P_0$ element does not verify the inf-sup condition, the eigenvalues computed by this method behave like those of P1 and P1* methods. In this case the first spurious eigenvalue converges to 18 (actually, 18 is a simple eigenvalue of the Dirichlet problem in the square, while in the numerical computation it is approximated by two distinct modes).

For the Neumann problem the following explicit formula of the numerical eigenvalues computed by $Q_1 - P_0$ element is available (see [4]):

$$\lambda_h^{mn} = (4/h^2)\frac{\sin^2(\frac{mh}{2}) + \sin^2(\frac{nh}{2}) - 2\sin^2(\frac{mh}{2})\sin^2(\frac{nh}{2})}{1-(2/3)(\sin^2(\frac{mh}{2}) + \sin^2(\frac{nh}{2})) + (4/9)\sin^2(\frac{mh}{2})\sin^2(\frac{nh}{2})}, \tag{4.1}$$

for $0 \le m, n \le N-1$, with $m+n \ne 0$ and $h = \pi/N$.

It is easy to verify that for m, n fixed $\lim_{h\to 0}\lambda_h^{mn} = m^2 + n^2 = \lambda^{mn}$, and hence $\lambda_h^{3\,3} \to 18$; on the other hand we also have by (4.1) $\lim_{h\to 0}\lambda_h^{N-1\,N-1} = 18$.

5. On the convergence of eigenvalues and eigenvectors

In this section, using the notation introduced in Section 2, we show that property (2.12) is a sufficient condition for the uniform convergence (2.11).

Theorem 5.1. *Condition (2.12) implies the uniform convergence (2.11).*

Proof. Let $f \in L^2(\Omega)$ be such that $||f||_0 = 1$. Since the eigenfunctions u_i, for $i \in \mathbb{N}$, are an orthonormal basis in $L^2(\Omega)$, we have

$$f = \sum_{i=1}^{\infty} \alpha_i u_i, \text{ where } \alpha_i = (f, u_i) \tag{5.1}$$

and

$$||f||_0^2 = \sum_{i=1}^{\infty} \alpha_i^2 = 1. \tag{5.2}$$

Let $I\!P_h$ be the $L^2(\Omega)$–projection operator defined in (2.14); then we can write

$$I\!P_h f = \sum_{i=1}^{N(h)} \alpha_i^h u_i^h, \text{ where } \alpha_i^h = (f, u_i^h) \tag{5.3}$$

and

$$||I\!P_h f||_0^2 = \sum_{i=1}^{N(h)} (\alpha_i^h)^2 \le ||f||_0^2 = 1. \tag{5.4}$$

Due to the definition of T_h, we have that $T_h f = T_h I\!P_h f$, so that we obtain

$$
\begin{aligned}
Tf - T_h f &= Tf - T_h I\!P_h f \\
&= T\left(\sum_{i=1}^{\infty} \alpha_i u_i\right) - T_h\left(\sum_{i=1}^{N(h)} \alpha_i^h u_i^h\right) \\
&= \sum_{i=1}^{\infty} \alpha_i T u_i - \sum_{i=1}^{N(h)} \alpha_i^h T_h u_i^h \\
&= \sum_{i=1}^{\infty} \frac{1}{\lambda_i}\alpha_i u_i - \sum_{i=1}^{N(h)} \alpha_i^h \frac{1}{\lambda_i^h} u_i^h .
\end{aligned}
\tag{5.5}
$$

For every $N \in \mathbb{N}$ we set $M = m(N)$ as defined in (2.7). Then we can write

$$
\begin{aligned}
Tf - T_h f &= \sum_{i=1}^{M} \frac{1}{\lambda_i}(\alpha_i u_i - \alpha_i^h u_i^h) \\
&\quad + \sum_{i=1}^{M} \left(\frac{1}{\lambda_i} - \frac{1}{\lambda_i^h}\right) \alpha_i^h u_i^h \\
&\quad - \sum_{i=M+1}^{N(h)} \frac{1}{\lambda_i^h}\alpha_i^h u_i^h + \sum_{i=M+1}^{\infty} \frac{1}{\lambda_i}\alpha_i u_i .
\end{aligned}
\tag{5.6}
$$

Fix now a positive ε. The last term is bounded in norm by $1/\lambda_{M+1}$ and is therefore smaller than ε for M big enough. The third term has a norm smaller than or equal to $1/\lambda_{M+1}^h$. For M fixed and h small enough, it will also be smaller than ε. The same is true for the first two terms: due to (2.12), for M fixed each one of them will have norm smaller than ε for h small enough, and the proof is complete. $\square$

Let us conclude this section showing that (2.11) is false for the third choice of spaces presented in the previous section.

Theorem 5.2. *Let Σ_h and V_h be defined as in (3.13). Then the sequence $\{T_h\}$ introduced in (2.8) does not converge to T in the norm of $\mathcal{L}(L^2(\Omega))$.*

Proof. In order to prove that (2.11) is false, we construct a sequence $\{v_h^*\} \subset V_h$ such that

$$
\begin{aligned}
&||v_h^*||_0 = 1 && \forall h > 0, \\
&||Tv_h^* - T_h v_h^*||_0 \not\to 0 && \text{as } h \to 0.
\end{aligned}
\tag{5.7}
$$

We take $v_h^* = \tilde{v}_h / ||\tilde{v}_h||_0$, where $\tilde{v}_h$ is defined in Remark 3.11. Hence (3.35) reduces to

$$
|(\operatorname{div} \underline{\tau}_h, v_h^*)| \le C ||\underline{\tau}_h||_0 \quad \forall \underline{\tau}_h \in \Sigma_h.
\tag{5.8}
$$

Since $||v_h^*||_0 = 1$ and v_h^* has zero mean-value in each macroelement, the sequence $\{v_h^*\}$ converges weakly to zero in $L^2(\Omega)$. Owing to the compactness of T, it follows that

$$
Tv_h^* \to 0 \quad \text{strongly in } L^2(\Omega).
\tag{5.9}
$$

Consider the solution $(\underline{\sigma}_h, u_h)$ of the problem

$$
\left\{
\begin{aligned}
&(\underline{\sigma}_h, \underline{\tau}_h) + (\operatorname{div} \underline{\tau}_h, u_h) = 0 && \forall \underline{\tau}_h \in \Sigma_h \\
&(\operatorname{div} \underline{\sigma}_h, v_h) = -(v_h^*, v_h) && \forall v_h \in V_h .
\end{aligned}
\right.
\tag{5.10}
$$

We observe that by definition $u_h = T_h v_h^*$. Our aim is to prove that $||u_h||_0 \not\to 0$.

From the second equation of (5.10) and the first of (5.7) we obtain

$$|(\operatorname{div}\underline{\sigma}_h, u_h)| = |(v_h^*, u_h)| \le ||u_h||_0, \tag{5.11}$$

and from the first equation of (5.10)

$$|(\operatorname{div}\underline{\sigma}_h, u_h)| = ||\underline{\sigma}_h||_0^2. \tag{5.12}$$

Using (5.8) and then the second equation of (5.10) we get

$$||\underline{\sigma}_h||_0 \ge \frac{1}{C}|(\operatorname{div}\underline{\sigma}_h, v_h^*)| = \frac{1}{C}(v_h^*, v_h^*) = \frac{1}{C}. \tag{5.13}$$

Finally, putting together (5.11), (5.12) and (5.13), we obtain

$$||u_h||_0 \ge \frac{1}{C^2}. \tag{5.14}$$

This concludes the proof of (5.7), because Tv_h^* tends to zero as $h \to 0$ while $u_h = T_h v_h^*$ does not. $\square$

Remark 5.3. A similar proof shows that (2.11) does not hold for the $Q_1 - P_0$ element of Subsection 3.2, nor for the $P_1^* - Q_0$ element of Subsection 3.4. On the other hand from (3.5) it follows that (2.11) holds for the mixed approach of Subsection 3.1.

6. Error estimates

The aim of this section is to recall, for convenience of the reader, the proof of the good behavior of the mixed approach described in Subsection 3.1 when applied to problem (2.1). In particular we shall prove the uniform convergence (2.11), together with the error estimates for eigenvalues and eigenvectors for a general choice of spaces Σ_h and V_h verifying some suitable abstract conditions. Results of this type are well-known. For instance the specific case of Raviart–Thomas elements can be found in [1] together with an abstract framework and several references.

We introduce the operator $S : L^2(\Omega) \to H(\operatorname{div};\Omega)$ given by

$$\begin{cases} (Sf, \underline{\tau}) + (\operatorname{div}\underline{\tau}, u) = 0 & \forall \underline{\tau} \in H(\operatorname{div};\Omega) \\ (\operatorname{div} Sf, v) = -(f, v) & \forall v \in L^2(\Omega) \end{cases} \tag{6.1}$$

and $\Sigma_0 = S(L^2(\Omega))$, which due to the regularity assumption on Ω verifies

$$\Sigma_0 \subset H^1(\Omega)^2. \tag{6.2}$$

Let us recall the so-called Fortin's operator (see [10]) $\Pi_h : \Sigma_0 \to \Sigma_h$

$$\begin{array}{ll} (\operatorname{div}(\underline{\sigma} - \Pi_h\underline{\sigma}), v_h) = 0 & \forall v_h \in V_h \\ ||\Pi_h\underline{\sigma}||_{\operatorname{div}} \le C||\underline{\sigma}||_1. & \end{array} \tag{6.3}$$

Proposition 6.1. *Let $f \in L^2(\Omega)$ be given. Suppose the existence of $\Pi_h : \Sigma_0 \to \Sigma_h$ verifying (6.3). Assume moreover the ellipticity in the kernel property (3.1). Then, using the notation of (2.3) and (2.8), the following estimates hold:*

$$\begin{array}{l} ||\underline{\sigma} - \underline{\sigma}_h||_0 \le C\left(||\underline{\sigma} - \Pi_h\underline{\sigma}||_0 + \frac{1}{\sqrt{\alpha}}\inf_{v_h\in V_h}||Tf - v_h||_0\right), \\ ||Tf - T_hf||_0 \le C\left(\inf_{v_h\in V_h}||Tf - v_h||_0 + ||\underline{\sigma} - \underline{\sigma}_h||_0\right). \end{array} \tag{6.4}$$

Proof. The result is essentially known (see e.g. [9, 12, 1, 8] for results of this type). However, for convenience of the reader, we give here the idea of the proof. In order to estimate the difference $||\mathbb{P}_hTf - T_hf||_0$ we can use the inf-sup condition which is implied by the existence of Π_h.

$$
\begin{aligned}
||\mathbb{P}_hTf - T_hf||_0 &\le C \sup_{\underline{\tau}\in\Sigma_h} \frac{(\mathbb{P}_hTf - T_hf, \operatorname{div}\underline{\tau})}{||\underline{\tau}||_{\operatorname{div}}} \\
&\le C \sup_{\underline{\tau}\in\Sigma_h} \frac{(\mathbb{P}_hTf - Tf, \operatorname{div}\underline{\tau}) + (Tf - T_hf, \operatorname{div}\underline{\tau})}{||\underline{\tau}||_{\operatorname{div}}} \\
&\le C||\mathbb{P}_hTf - Tf||_0 + \sup_{\underline{\tau}\in\Sigma_h} \frac{-(\underline{\sigma} - \underline{\sigma}_h, \underline{\tau})}{||\underline{\tau}||_{\operatorname{div}}} \\
&\le C||\mathbb{P}_hTf - Tf||_0 + ||\underline{\sigma} - \underline{\sigma}_h||_0.
\end{aligned} \tag{6.5}
$$

The second estimate of (6.4) is then obtained by the triangular inequality. Finally, the first one can be easily deduced using again the triangular inequality, the error equations and the ellipticity in the kernel property (3.1):

$$
\begin{aligned}
||\Pi_h\underline{\sigma} - \underline{\sigma}_h||_0^2 &= (\Pi_h\underline{\sigma} - \underline{\sigma}, \Pi_h\underline{\sigma} - \underline{\sigma}_h) + (\underline{\sigma} - \underline{\sigma}_h, \Pi_h\underline{\sigma} - \underline{\sigma}_h) \\
&= (\Pi_h\underline{\sigma} - \underline{\sigma}, \Pi_h\underline{\sigma} - \underline{\sigma}_h) - (\operatorname{div}(\Pi_h\underline{\sigma} - \underline{\sigma}_h), Tf - \mathbb{P}_hTf) \\
&\le ||\Pi_h\underline{\sigma} - \underline{\sigma}_h||_0||\underline{\sigma} - \Pi_h\underline{\sigma}||_0 + ||\operatorname{div}(\Pi_h\underline{\sigma} - \underline{\sigma}_h)||_0||Tf - \mathbb{P}_hTf||_0 \\
&\le ||\Pi_h\underline{\sigma} - \underline{\sigma}_h||_0 \left(||\underline{\sigma} - \Pi_h\underline{\sigma}||_0 + \tfrac{1}{\sqrt{\alpha}}||Tf - \mathbb{P}_hTf||_0\right).
\end{aligned} \tag{6.6}
$$

□

Remark 6.1. It is well-known that the existence of the operator Π_h verifying (6.3) together with (3.1) implies the following error estimate for problem (2.8)

$$
||\underline{\sigma} - \underline{\sigma}_h||_{\operatorname{div}} \le C\left(\inf_{v_h\in V_h} ||u - v_h||_0 + \inf_{\underline{\tau}_h\in\Sigma_h} ||\underline{\sigma} - \underline{\tau}_h||_{\operatorname{div}}\right). \tag{6.7}
$$

Actually, estimates (6.4) and (6.7) are not enough to ensure the uniform convergence (2.11). This has been proved with the counter-example given in the previous section. The $P_1 - \operatorname{div} P_1$ element and the $P_1^* - Q_0$ element on a criss-cross mesh verify both (3.1) and (3.2) hypotheses. Moreover, it is not difficult to show the existence of an operator Π_h which verifies the Fortin's hypothesis (6.3). However they do not verify the uniform convergence (2.11), hence they are not well-suited for the approximation of eigenproblem (2.1), as it has been proved in the previous section and numerically demonstrated in Section 4.

From Proposition 6.1 it follows that it will be sufficient to add the following hypothesis for the uniform convergence (2.11)

$$
||I - \Pi_h||_{\mathcal{L}(\Sigma_0, L^2(\Omega)^2)} \to 0. \tag{6.8}
$$

The following theorem gives the error estimates for eigenproblem (2.2).

Theorem 6.2. *Assume that there exists a linear operator* $\Pi_h : \Sigma_0 \to \Sigma_h$ *which satisfies the* Fortin's conditions *(6.3) and (6.8).*

Assume also the ellipticity in the kernel *property (3.1).*

For every $N \in \mathbb{N}$ *define moreover the following function* $\rho_N :]0,1] \to \mathbb{R}$

$$
\rho_N(h) = \sup_{u\in\oplus_{i=1}^{m(N)}E_i} \left(\inf_{v_h\in V_h} ||u - v_h||_0 + ||\underline{\nabla}u - \Pi_h\underline{\nabla}u||_0\right). \tag{6.9}
$$

Then problem (2.2) is well-posed and the following error estimates hold true with C independent of h:

$$\begin{aligned}\sum_{i=1}^{m(N)} |\lambda_i - \lambda_i^h| \le C(\rho_N(h))^2, \\ \hat{\delta}\left(\oplus_{i=1}^{m(N)} E_i, \oplus_{i=1}^{m(N)} E_i^h\right) \le C\rho_N(h).\end{aligned} \tag{6.10}$$

Proof. The proof is an immediate consequence of estimate (6.4) of Proposition 6.1, the definition (6.9) of ρ_N and classical results on eigenvalues approximation (see [15], (3.17), (3.18) for the derivation of estimates (6.10) and the references therein). □

Remark 6.3. This last theorem implies, in particular, that the mixed spaces recalled in Section 3 give good results for the approximation of problem (2.1). For instance, when using the RT elements of lowest degree it is well-known that for N fixed one has $\rho_N(h) = O(h)$ (as it is also easy to check using (3.4) and (6.9)).

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