Physica D 240 (2011) 754–766

Contents lists available at ScienceDirect

Physica D

journal homepage: www.elsevier.com/locate/physd

A Cahn–Hilliard model in a domain with non-permeable walls

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ARTICLE INFO

Article history:
Received 23 July 2010
Received in revised form
13 December 2010
Accepted 15 December 2010
Available online 22 December 2010
Communicated by R. Temam

Keywords:
Cahn–Hilliard equation
Dynamic boundary conditions
Weak solutions
Parabolic regularization
Global attractors
ω-limit sets

ABSTRACT

We consider in this article a Cahn–Hilliard model in a bounded domain with non-permeable walls, characterized by dynamic-type boundary conditions. Dynamic boundary conditions for the Cahn–Hilliard system have recently been proposed by physicists in order to account for the interactions with the walls in confined systems and are obtained by writing that the total bulk mass is conserved and that there is a relaxation dynamics on the boundary. However, in the case of non-permeable walls, one should also expect some mass on the boundary. It thus seems more realistic to assume that the total mass, in the bulk and on the boundary, is conserved, which leads to boundary conditions of a different type. For the resulting mathematical model, we prove the existence and uniqueness of weak solutions and study their asymptotic behavior as time goes to infinity.

1. Introduction

We consider in this article the following Cahn–Hilliard system describing the evolution of the order parameter of a binary material associated with a particular class of *dynamic boundary conditions*:

$$\rho_t - \Delta\mu = 0, \quad \text{in } \Omega, \tag{1.1}$$

$$\mu = -\Delta\rho + f(\rho), \quad \text{in } \Omega, \tag{1.2}$$

$$w\rho_t - \delta\Delta_\Gamma\mu = -\partial_{\boldsymbol{n}}\mu, \quad \text{on } \Gamma, \tag{1.3}$$

$$w\mu = -\sigma\Delta_\Gamma\rho + g(\rho) + \partial_{\boldsymbol{n}}\rho, \quad \text{on } \Gamma. \tag{1.4}$$

A physical derivation of this model, based on a variational principle, is detailed in the next section. Here, $\Omega \subset \mathbb{R}^3$ is a smooth and bounded domain with boundary Γ corresponding to the binary material under consideration. Moreover, δ and σ are nonnegative parameters (which can vanish, not necessarily simultaneously, in the absence of boundary diffusion), w is a bounded and nondegenerate weight function, Δ_Γ is the Laplace–Beltrami operator, and the nonlinear function f is the derivative of a λ-convex and possibly *singular* "bulk" potential F. Finally, g is a smooth function with controlled growth at infinity. The terminology "singular potential" (cf. Remark 3.1 below) refers to the fact that F is allowed to take identically the value $+\infty$ outside a bounded interval $I \subset \mathbb{R}$, while λ-convex means convex up to a quadratic perturbation.

The Cahn–Hilliard system describes important qualitative features of two-phase systems related to phase separation processes. This phenomenon can be observed, e.g., when a binary alloy is cooled down sufficiently. One then observes a partial nucleation (i.e., the appearance of nucleides in the material) or a total nucleation, the so-called spinodal decomposition: the material quickly becomes inhomogeneous, forming a fine-grained structure in which each of the two components appears more or less alternatively. In a second stage, which is called coarsening and which occurs at a slower time scale and is less understood, these microstructures coarsen. Such phenomena play an essential role in the mechanical properties of the material, e.g., strength. We refer the reader to, e.g., [1–3] for more details.

Dynamic boundary conditions have recently been proposed by physicists, in the context of the Cahn–Hilliard equation, in order to account for the interactions with the walls in confined systems (cf. [4–6] and the references therein). In particular, such boundary conditions have mainly been studied for polymer mixtures (although this should also be important in other systems, such as binary metallic alloys): from a technological point of view, binary polymer mixtures are particularly interesting, since the structures occurring during the phase separation process may be frozen by a rapid quench into the glassy state; microstructures at surfaces on very small length scales can be produced in this way.

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doi:10.1016/j.physd.2010.12.007

More precisely, the following boundary conditions have been proposed:

$$\partial_{\mathbf{n}}\mu = 0, \quad \text{on } \Gamma, \tag{1.5}$$

$$\rho_t - \sigma\Delta_\Gamma\rho + g(\rho) + \partial_{\mathbf{n}}\rho = 0, \quad \text{on } \Gamma,\ \sigma > 0, \tag{1.6}$$

the second boundary condition being called a dynamic boundary condition, in the sense that the kinetics, i.e., ρ_t, appears explicitly. The Cahn–Hilliard system, endowed with these boundary conditions, has been studied in [7–13] (see also [14–20] for similar boundary conditions for the Caginalp phase-field system).

Now, the first boundary condition, namely, $\partial_{\mathbf{n}}\mu = 0$, on Γ, means that there is no mass flux on the boundary (the second one being obtained by assuming a relaxation dynamics on the boundary and yielding that the total free energy decreases) and implies, integrating (formally) (1.1) over Ω, the conservation of the total mass in Ω. However, it also seems reasonable, to fully account for the interactions of the material's components with the walls, to actually write that the total mass, in Ω and on Γ, is conserved; indeed, it seems reasonable to assume that there is also some mass on the boundary. This leads (see the next section for details) to the boundary conditions (1.3)–(1.4).

Our first aim in the present article is to give a physical derivation of system (1.1)–(1.4) on the basis of a variational principle. Actually, similar boundary conditions (with $\delta = 0$, w constant and g linear) have been derived in [21], though in a different context, namely, in the case of permeable walls, and studied in [21–24] for a regular nonlinear term f.

Then, we analyze, from a mathematical point of view, the well-posedness issue for this model and try to deal with the most general conditions on the nonlinear terms. In particular, in order to treat singular functions f, we are forced to set the problem in a suitable weak framework by considering duality techniques in negative order Sobolev spaces. Indeed, it is expected here that, when the bulk potential F is bounded ($F' = f$), we can have nonexistence of classical (i.e., in the sense of distributions) solutions. This can be seen, as in [11] for the usual dynamic boundary conditions, by considering the scalar ODE

$$y'' - f(y) = 0, \quad x \in (-1, 1), \qquad y'(\pm 1) = C > 0,$$

for large C's (which corresponds to the one-dimensional stationary problem with $\mu = 0$ and $g \equiv -C$; note that, in one space dimension, the Laplace–Beltrami operator does not make sense and does not appear in the boundary conditions). This type of approach, based on duality, has been proved to be effective for other similar models (see, e.g., [20,25]; see also [11] for a different, though related, notion of a weak solution, based on variational inequalities).

Once the well-posedness of the system has been established in the proper weak setting, we discuss further properties of the solutions, such as parabolic regularization effects, dissipativity, and long-time behavior. In particular, we show that, if the functions f and g satisfy suitable growth conditions (which, in particular, restrict the class of admissible functions f in the singular case), then the solutions satisfy instantaneous regularization properties and can be intended in a stronger sense (i.e., pointwise) for all strictly positive times (for all times if the initial datum is also more regular). We also prove the existence of a compact global attractor for the dynamical process generated by the system. Finally, we show the existence of ω-limit sets of solution trajectories and prove that they only consist of stationary states. We also prove that, if f and g are analytic functions and some other technical conditions hold, then the ω-limit set of any trajectory consists of one single point. This result is shown by appealing to the well-known Łojasiewicz–Simon method (cf., e.g., [7]). In the case of a singular function f, the key point in the proof consists in showing that any solution is, at least for large times, uniformly separated from the singular points of f (i.e., from the endpoints of I).

This article is organized as follows. In Section 2, we give a derivation of (1.3)–(1.4). Then, in Section 3, we give our main assumptions and state and prove our results.

2. Derivation of the model

First, we set $H := L^2(\Omega)$ and denote by $(\cdot,\cdot)$ the scalar product both in H and in H^3 and by $\|\cdot\|$ the related norm. Moreover, we set $V := H^1(\Omega)$ and denote by V' the (topological) dual of V. The duality between V' and V is indicated by $\langle\cdot,\cdot\rangle$.

We also set $H_\Gamma := L^2(\Gamma)$ and $V_\Gamma := H^1(\Gamma)$ and denote by $(\cdot,\cdot)_\Gamma$ the standard scalar product in H_Γ, by $\|\cdot\|_\Gamma$ the corresponding norm, and by $\langle\cdot,\cdot\rangle_\Gamma$ the duality between V'_Γ and V_Γ. In general, $\|\cdot\|_X$ indicates the norm in the generic (real) Banach space X and $\langle\cdot,\cdot\rangle_X$ stands for the duality between X' and X.

We can then define the Hilbert spaces

$$\mathcal{H} := H \times H_\Gamma \quad \text{and} \quad \mathcal{V} := \{z \in V : z|_\Gamma \in V_\Gamma\}, \tag{2.1}$$

endowed with the natural scalar products and norms. Unless otherwise specified, in what follows, we will impose the following convention: when we write $h \in \mathcal{H}$, h will be interpreted as a pair of functions belonging, respectively, to H and to H_Γ and both denoted by the same letter h. Analogously, when we consider $v \in \mathcal{V}$ (or even $v \in V$), the symbol v will be intended, depending on the context, either as a function defined in Ω, or as a pair formed by a function in Ω and its trace on Γ.

We will also need some weighted spaces. We take

$$w \in L^\infty(\Gamma), \qquad 0 < w_* \le w(x) \le w^* \quad \text{for a.e. } x \in \Gamma, \tag{2.2}$$

where w_*, w^* are given constants. Then, we introduce, on $\overline{\Omega} = \Omega \cup \Gamma$, the measure d$m$ given by

$$\int_{\overline{\Omega}} v\mathrm{d}m := \int_\Omega v\mathrm{d}x + \int_\Gamma vw\,\mathrm{d}\Sigma, \tag{2.3}$$

where v represents a generic function in $L^1(\Omega) \times L^1(\Gamma)$. We also define

$$m(v) := \left(\int_\Omega v\mathrm{d}x + \int_\Gamma vw\,\mathrm{d}\Sigma\right)\left(|\Omega| + \int_\Gamma w\,\mathrm{d}\Sigma\right)^{-1}, \tag{2.4}$$

i.e., the average of v with respect to the measure dm.

Thanks to (2.2), dm is equivalent to the Lebesgue measure d$x \otimes$ dΣ. In particular, the scalar product

$$(h,k)_m := \int_{\overline{\Omega}} hk\mathrm{d}m = \int_\Omega hk\mathrm{d}x + \int_\Gamma hkw\,\mathrm{d}\Sigma \tag{2.5}$$

generates on $\mathcal{H}$ a norm $\|\cdot\|_m$ which is equivalent to the standard norm $\|\cdot\|_{\mathcal{H}}$.

We can now provide a mathematically accurate derivation of system (1.1)–(1.4), starting from mechanical principles.

First, we assume the validity of Eq. (1.1) which describes the balance of mass in Ω. Indeed, integrating it over Ω, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho\mathrm{d}x = \int_\Gamma \partial_{\mathbf{n}}\mu\,\mathrm{d}\Sigma \tag{2.6}$$

and we have to specify the evolution of μ. To this end, we define the *free energy* of the system as

$$\mathcal{E}(\rho) := \int_\Omega \left(\frac{|\nabla\rho|^2}{2} + F(\rho)\right)\mathrm{d}x + \int_\Gamma \left(\frac{\sigma}{2}|\nabla_\Gamma\rho|^2 + G(\rho)\right)\mathrm{d}\Sigma, \tag{2.7}$$

i.e., the sum of the Ginzburg–Landau (bulk) free energy and of a surface free energy, where ∇_Γ is the tangential gradient on Γ and σ may vanish, due to a possible lack of boundary diffusive effects. Here, F and G are suitable antiderivatives of the functions f and g in (1.2) and (1.4).

We then assume that, a.e. in the reference time interval $(0, T)$,

$$\mu = \partial_m \mathcal{E}(\rho), \tag{2.8}$$

∂_m denoting the subdifferential in the space $\mathcal{H}$ with respect to the scalar product (2.5). As we will see below, by introducing the weight m, we are able to consider the situation when the "boundary mass" is linked to the variable ρ in a different way with respect to the "bulk mass". For instance, one can think of the case when the dynamic boundary conditions arise as an approximation of a thin diffusive layer occupied by a different material, as in the so-called "concentrated capacity" models; see, e.g., [26–29].

It is then not difficult to check that (2.8) can be rephrased as the system

$$\mu = -\Delta\rho + f(\rho), \quad \text{in } \Omega, \tag{2.9}$$

$$\mu = \frac{1}{w}(-\sigma\Delta_\Gamma\rho + g(\rho) + \partial_{\boldsymbol{n}}\rho), \quad \text{on } \Gamma, \tag{2.10}$$

where Δ_Γ is the Laplace–Beltrami operator. Actually, to see that (2.9)–(2.10) is equivalent to (2.8), it is sufficient to take a sufficiently regular test function z, multiply (2.9) by $z-\rho$ and (2.10) by $w(z-\rho)$, and integrate (i.e., we test (2.9)–(2.10) with the test function $z-\rho$ for the scalar product (2.5)). Integrations by parts then lead to

$$(\mu, z-\rho)_m \le \mathcal{E}(z) - \mathcal{E}(\rho), \tag{2.11}$$

which coincides with (2.8) by the definition of subdifferentials.

Of course, coupling (1.1) with (2.9)–(2.10) does not give a closed system, since a boundary condition for μ is still missing (while (2.10) or, equivalently, (1.4) provides, in fact, a boundary condition for ρ).

The main novelty of the present approach is to consider, as mentioned in the introduction, the class of those boundary conditions which ensure the conservation of the *total* (i.e., bulk plus boundary) mass. More precisely, we ask that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\overline{\Omega}} \rho \mathrm{d}m = \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega \rho \mathrm{d}x + \int_\Gamma \rho w \,\mathrm{d}\Sigma\right) = 0. \tag{2.12}$$

This is in contrast with what happens with the usual Cahn–Hilliard models for which only the bulk mass is conserved. However, in the framework of dynamic boundary problems, it seems reasonable to assume that the boundary gives a non-negligible contribution to diffusion. Hence, (2.12) may appear as a more realistic condition.

Then, using (2.6), (2.12) yields the *compatibility condition*

$$\int_\Gamma (\rho_t w + \partial_{\boldsymbol{n}}\mu)\,\mathrm{d}\Sigma = 0. \tag{2.13}$$

A class of boundary conditions guaranteeing the validity of (2.13) (and, hence, of (2.12)) is given by

$$\rho_t + \frac{1}{w}(-\delta\Delta_\Gamma\mu + \partial_{\boldsymbol{n}}\mu) = 0, \quad \text{on } \Gamma, \tag{2.14}$$

where $\delta \ge 0$. Of course, (2.14) can be rephrased as the equivalent form (1.3).

Remark 2.1. In [21], as mentioned in the introduction, similar boundary conditions are derived in the context of permeable walls. Compared with our derivation, (2.14) (with $w \equiv$ Const. and $\delta = 0$; such a boundary condition is then called a Wentzell boundary condition) is somehow assumed and yields the conservation of mass (2.12), whereas, here, (2.8) (and, thus, (2.10)) is assumed.

3. Weak formulation and main results

We introduce here our main assumptions, together with a number of tools which are needed in order to reformulate system (1.1)–(1.4) in a mathematically precise way and give a rigorous statement of our results.

We first need some further discussion on the functional spaces. Actually, we notice that we have the chain of continuous embeddings

$$\mathcal{V} \subset V \subset \mathcal{H} \subset V' \subset \mathcal{V}'. \tag{3.1}$$

More precisely, identifying $\mathcal{H}$ with its dual through the scalar product (2.5), we obtain the Hilbert triplets $(V, \mathcal{H}, V')$ and $(\mathcal{V}, \mathcal{H}, \mathcal{V}')$. Indeed, it is not difficult to prove that the space $\mathcal{V}$ is dense in $\mathcal{H}$.

For later convenience, we set, for $\eta \ge 0$,

$$V_\eta := \mathcal{V} \quad \text{if } \eta > 0 \quad \text{and} \quad V_\eta := V \quad \text{if } \eta = 0 \tag{3.2}$$

(in applications, $\eta = \delta$ or σ). In particular, the norm $\|\cdot\|_{V_\eta}$ coincides with the norm of V for $\eta = 0$ and with that of $\mathcal{V}$ for $\eta > 0$.

We also define the elliptic operators

$$A : V \to V', \qquad \langle Az_1, z_2\rangle := \int_\Omega \nabla z_1 \cdot \nabla z_2, \tag{3.3}$$

$$A_\Gamma : V_\Gamma \to V_\Gamma', \qquad \langle A_\Gamma z_1, z_2\rangle_\Gamma := \int_\Gamma \nabla_\Gamma z_1 \cdot \nabla_\Gamma z_2, \tag{3.4}$$

$$\mathcal{A} : \mathcal{V} \to \mathcal{V}', \qquad \langle \mathcal{A}z_1, z_2\rangle_{\mathcal{V}} := \langle Az_1, z_2\rangle + \langle A_\Gamma z_1, z_2\rangle_\Gamma, \tag{3.5}$$

$$\begin{aligned} &\mathcal{A}_\eta : V_\eta \to V_\eta', \\ &\langle \mathcal{A}_\eta z_1, z_2\rangle_{V_\eta} := \langle Az_1, z_2\rangle + \eta\langle A_\Gamma z_1, z_2\rangle_\Gamma. \end{aligned} \tag{3.6}$$

Let us now present our basic hypotheses on the nonlinear terms. We assume that I is an open, possibly bounded, interval of $\mathbb{R}$ containing 0 and that

$$f \in C^{0,1}_{\mathrm{loc}}(I, \mathbb{R}), \qquad g \in C^{0,1}_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}), \qquad f(0) = g(0) = 0. \tag{fg1}$$

Moreover, we assume that

$$\lim_{r\to\partial I} f(r)\operatorname{sign} r = \lim_{|r|\to+\infty} g(r)\operatorname{sign} r = +\infty, \tag{fg2}$$

$$f'(r), g'(r) \ge -\lambda \quad \text{for some } \lambda \ge 0 \tag{fg3}$$

and for a.e. r in I and in $\mathbb{R}$, respectively. Then, we define, whenever they make sense, the *potentials*

$$F(r) := \int_0^r f(s)\,\mathrm{d}s, \qquad G(r) := \int_0^r g(s)\,\mathrm{d}s \tag{3.7}$$

and it follows from (fg2) that both F and G are bounded from below. If $I \ne \mathbb{R}$, F is extended by continuity to the closure $\bar{I}$ and then extended by $+\infty$ outside $\bar{I}$.

Remark 3.1. Actually, we will speak of *singular* potentials F in the case when I is a bounded interval. A physically relevant case is given by the so-called logarithmic potential

$$F(r) = (1+r)\ln(1+r) + (1-r)\ln(1-r) - \lambda r^2, \quad \lambda \ge 0, \tag{3.8}$$

where $I = (-1, 1)$, of course. The recent literature on Cahn–Hilliard problems (see, e.g., [30,11]) shows that singular potentials are considerably more difficult to deal with than "regular" ones, especially in connection with dynamic boundary conditions.

To deduce a weak formulation of (1.1)–(1.4), we multiply (1.2) and (1.4) by a test function $\phi \in V_\sigma$ and (1.1) and (1.3) by $\psi \in V_\delta$ and then integrate. This leads to the relations

$$\int_\Omega \mu\phi + \int_\Gamma \mu\phi w = \int_\Omega \nabla\rho\cdot\nabla\phi + \int_\Omega f(\rho)\phi + \sigma\int_\Gamma \nabla_\Gamma\rho\cdot\nabla_\Gamma\phi + \int_\Gamma g(\rho)\phi, \tag{3.9}$$

$$\int_\Omega \rho_t\psi + \int_\Gamma \rho_t\psi w = -\int_\Omega \nabla\mu\cdot\nabla\psi - \delta\int_\Gamma \nabla_\Gamma\mu\cdot\nabla_\Gamma\psi. \tag{3.10}$$

We also set

$$\mathcal{F}:\mathcal{H}\to\mathcal{H},\qquad \mathcal{F}(v):=f(v),\quad \text{in }\Omega,\qquad \mathcal{F}(v):=\frac{g(v)}{w},\quad \text{on }\Gamma, \tag{3.11}$$

$$\mathcal{B}:\mathcal{H}\to\mathcal{H},\qquad \mathcal{B}(v):=f(v)+\lambda v,\quad \text{in }\Omega,\qquad \mathcal{B}(v):=\frac{g(v)+\lambda v}{w},\quad \text{on }\Gamma. \tag{3.12}$$

Actually, the operator $\mathcal{F}$ is seen as a linear perturbation of the maximal monotone operator $\mathcal{B}$ acting from $\mathcal{H}$ to itself. Notice the occurrence of the function w in the denominator, due to the fact that $\mathcal{H}$ is identified with $\mathcal{H}'$ through the scalar product (2.5), in relation with the way the Hilbert triplet $(\mathcal{V},\mathcal{H},\mathcal{V}')$ is built. Of course, the effective domains of $\mathcal{B}$ and $\mathcal{F}$ are in general strict subsets of $\mathcal{H}$.

Using (3.3)–(3.6) and (3.11), system (3.9)–(3.10) can then be rewritten in the more compact form

$$\rho_t = -\mathcal{A}_\delta\mu,\quad \text{in } V_\delta', \tag{3.13}$$

$$\mu = \mathcal{A}_\sigma\rho + \mathcal{F}(\rho),\quad \text{in } V_\sigma'. \tag{3.14}$$

However, under general conditions on the functions f, g, it is very difficult, if not impossible, to prove an existence result for system (3.13)–(3.14) in its present form. More precisely, due to the dynamic boundary conditions, an $\mathcal{H}$-uniform bound on the term $\mathcal{F}(\rho)$ is generally out of reach. One way to overcome this difficulty is to introduce a weaker notion of a solution, by suitably relaxing the functional $\mathcal{F}$ (a different, albeit closely related, approach has been proposed in [11]).

To this end, we first define the *convex* (cf. (fg3)) functional

$$J:\mathcal{H}\to\mathbb{R}\cup\{+\infty\},\qquad J(\rho):=\int_\Omega\left(F(\rho_1)+\frac{\lambda}{2}\rho_1^2\right)+\int_\Gamma\left(G(\rho_2)+\frac{\lambda}{2}\rho_2^2\right), \tag{3.15}$$

where, here, $\rho=(\rho_1,\rho_2)$. Furthermore, for $\rho=(\rho_1,\rho_2)\in\mathcal{H}$, we say that $\rho\in\mathcal{D}(J)$ if and only if $\rho_1\in L^2(\Omega)$, $\rho_2\in L^2(\Gamma)$, $F(\rho_1)\in L^1(\Omega)$ and $G(\rho_2)\in L^1(\Gamma)$. If $\rho\in\mathcal{H}\setminus\mathcal{D}(J)$, then we set $J(\rho)=+\infty$. Then, J is proper, convex, and lower semicontinuous on $\mathcal{H}$. Notice that, for $\rho,\kappa\in\mathcal{D}(J)$, it holds that

$$\kappa=(\kappa_1,\kappa_2)\in\partial_m J(\rho)\iff \kappa_1=f(\rho_1)+\lambda\rho_1\quad\text{a.e. in }\Omega\quad \text{and}\quad \kappa_2=\frac{g(\rho_2)+\lambda\rho_2}{w}\quad\text{a.e. on }\Gamma. \tag{3.16}$$

In other words, $\mathcal{B}$ coincides with the $\mathcal{H}$-subdifferential of J with respect to the scalar product (2.5). Note that, if $\kappa=(\kappa_1,\kappa_2)\in V_\sigma$, then $\kappa_1=\kappa|_\Omega$ and $\kappa_2=\kappa|_\Gamma$. Then, since $V_\sigma\subset\mathcal{H}$ for any $\sigma\geq 0$, we can consider the restriction of J to V_σ and denote by $\mathcal{B}_{\rm w}$ (i.e., the "weak" form of $\mathcal{B}$) its subdifferential with respect to the duality between V_σ and V_σ'. More precisely, for $\rho\in V_\sigma$ and $h\in V_\sigma'$, we have

$$h\in\mathcal{B}_{\rm w}(\rho)\overset{\rm def}{\iff}\langle h,z-\rho\rangle_{V_\sigma}\leq J(z)-J(\rho)\quad\forall z\in V_\sigma. \tag{3.17}$$

It is not difficult to prove that, considering the Hilbert triplet $V_\sigma\subset\mathcal{H}\subset V_\sigma'$, graph$(\mathcal{B})\cap(V_\sigma\times V_\sigma')$ is included in graph$(\mathcal{B}_{\rm w})$. In other words,

$$\partial_m J(z)\subset\mathcal{B}_{\rm w}(z)\quad\forall z\in V_\sigma. \tag{3.18}$$

However, it is reasonable to expect this inclusion to be strict, both because the domain of $\mathcal{B}_{\rm w}$ may be larger than the restriction to V_σ of the domain of $\mathcal{B}$ and because $\mathcal{B}_{\rm w}$ is expected to be a *multivalued* operator in the case when f is singular (for more details, we refer the reader to [31,32], where this type of relaxed operators is deeply examined when V_σ is replaced by $W_0^{1,p}(\Omega)$).

Introducing also the linear and continuous operator

$$\mathcal{I}_{\rm w}:V_\sigma\to V_\sigma',\qquad \langle\mathcal{I}_{\rm w}z_1,z_2\rangle_{V_\sigma}:=\int_\Omega z_1z_2+\int_\Gamma z_1z_2, \tag{3.19}$$

and defining $\mathcal{F}_{\rm w}:=\mathcal{B}_{\rm w}-\lambda\mathcal{I}_{\rm w}$, we can reformulate system (3.13)–(3.14) in a weak setting and state a related well-posedness result.

Theorem 3.2. *Let us assume that* (2.2) *and* (fg1)–(fg3) *hold and that*

$$\rho_0\in V_\sigma,\quad F(\rho_0)\in L^1(\Omega),\qquad G(\rho_0)\in L^1(\Gamma). \tag{3.20}$$

Assume also that

$$m_0:=m(\rho_0)\in I \tag{3.21}$$

and take some $T>0$ as a final time. Then, there exist functions ρ and μ having the regularity

$$\rho\in L^\infty(0,T;V_\sigma),\qquad \rho_t\in L^2(0,T;V_\delta'), \tag{3.22}$$

$$\mu\in L^2(0,T;V_\delta) \tag{3.23}$$

which satisfy, a.e. in $(0,T)$,

$$\rho_t=-\mathcal{A}_\delta\mu,\quad \text{in } V_\delta', \tag{3.24}$$

$$\mu\in\mathcal{A}_\sigma\rho+\mathcal{F}_{\rm w}(\rho),\quad \text{in } V_\sigma', \tag{3.25}$$

as well as the initial condition

$$\rho|_{t=0}=\rho_0. \tag{3.26}$$

Moreover, the function ρ is unique and we have, in particular, the contractive estimate

$$\|\rho_1-\rho_2\|_{L^2(0,T;V_\sigma)}\leq c(T)\|\rho_{0,1}-\rho_{0,2}\|_{V_\delta'}, \tag{3.27}$$

where $\rho_{0,1}$ and $\rho_{0,2}$ are two initial data satisfying (3.20)–(3.21) *for the same value m_0 and ρ_1 and ρ_2 are the ρ-components of the corresponding solutions.*

Notice the occurrence of the inclusion sign in (3.25) which is due to the possible multivalued character of $\mathcal{B}_{\rm w}$ (and, hence, of $\mathcal{F}_{\rm w}$).

Remark 3.3. Condition (3.21) is not automatic (i.e., it does not always follow from (3.20)). For instance, in the case of the logarithmic potential (3.8), we can have $\rho_0\equiv 1$ and $F(\rho_0)\in L^1(\Omega)$. However, we cannot deal with such an initial datum, since it does not fulfill (3.21).

Remark 3.4. Notice that the above theorem only states the uniqueness of ρ. Instead, μ may possibly be nonunique. Actually, this a standard issue in Cahn–Hilliard models when a multivalued operator appears in the formulation of the system (as happens in our case with $\mathcal{F}_{\rm w}$). A specific example for which nonuniqueness occurs is constructed in [33, Rem. 2.3].

Our next result is devoted to the dissipativity of the system and to smoothing properties of the solutions. Actually, in view of a long-time analysis, we introduce the "finite-energy" phase space

$$\mathcal{X}:=\{\rho\in V_\sigma:\mathcal{E}(\rho)<+\infty\}. \tag{3.28}$$

Notice that only the component ρ of a solution is considered, due to the possible nonuniqueness of μ. The space $\mathcal{X}$ can be endowed with the natural distance

$$\mathrm{dist}_{\mathcal{X}}(\rho_1,\rho_2) := \|\rho_1-\rho_2\|_{V_\sigma} + \|F(\rho_1)-F(\rho_2)\|_{L^1(\Omega)} + \|G(\rho_1)-G(\rho_2)\|_{L^1(\Gamma)}. \tag{3.29}$$

It is then not difficult to prove (cf., e.g., [34, Lemma 3.8]) that $(\mathcal{X}, \mathrm{dist}_{\mathcal{X}})$ is a *complete* metric space. To account for the conservation of the spatial average (cf. (2.12) and (3.21)), given $m_0 \in I = \mathrm{dom}(f)$, we also set

$$\mathcal{X}_0 := \{\rho \in \mathcal{X} : m(\rho) = m_0\}. \tag{3.30}$$

Theorem 3.5. *Let the assumptions of Theorem 3.2 hold. In particular, let us assume that, for a given m_0 as above, $\rho_0 \in \mathcal{X}_0$ and let us set $\mathbb{E}_0 := \mathcal{E}(\rho_0)$. Then, for all $t \geq 0$, $\rho(t) \in \mathcal{X}_0$. Moreover, there exist a number R_0 which is independent of the initial datum (but may depend on m_0) and a time T_0 depending only on $\mathbb{E}_0$ such that $\mathcal{E}(\rho(t)) \leq R_0$ for all $t \geq T_0$. In other words, the set $B_0 := \{\rho \in \mathcal{X}_0 : \mathcal{E}(\rho) \leq R_0\}$ is an absorbing set for the dynamical system $S(\cdot)$ associated with (the ρ-components of) the solutions of system (3.24)–(3.25).*

As a next step, without assuming any further property on the initial datum, we show that any weak solution satisfies some instantaneous regularization effect.

Theorem 3.6. *Let the assumptions of Theorem 3.5 hold and let, as above, $\mathbb{E}_0 := \mathcal{E}(\rho_0)$. Let (ρ, μ) be a weak solution. Then, for any $\tau > 0$, the following properties hold:*

$$\|\rho_t\|_{L^2(t,t+1;V_\sigma)} \leq Q(\tau^{-1}, \mathbb{E}_0) \quad \forall t \geq \tau, \tag{3.31}$$

$$\|\mu\|_{L^\infty(\tau,+\infty;V_\delta)} \leq Q(\tau^{-1}, \mathbb{E}_0), \tag{3.32}$$

where Q is a computable nonnegative function and is monotone increasing with respect to each argument.

Restricting the class of admissible nonlinearities, we can show that properties (3.31)–(3.32) also entail further regularity for ρ, both in Ω and on Γ.

Theorem 3.7. *Let the assumptions of Theorem 3.5 hold and let, in addition, for a.e. $r \in I$,*

$$|f'(r)| \leq c_f(1+|f(r)|^{p_0}) \quad \text{for some } p_0 \in (0,2) \text{ if } \delta > 0, \tag{3.33}$$

$$|f'(r)| \leq c_f(1+|f(r)|^{4/3}) \quad \text{if } \delta = 0 \tag{3.34}$$

for some constant $c_f > 0$. Let us also assume that

$$\text{if } I = \mathbb{R}, \quad \text{then } \exists c_{f,g} > 0 : |g(r)| \leq c_{f,g}(1+|f(r)|) \quad \forall r \in \mathbb{R}. \tag{3.35}$$

Let (ρ, μ) be a weak solution. Then, for any $\tau > 0$,

$$\|\rho\|_{L^\infty(\tau,+\infty;H^{3/2}(\Omega))} + \sigma^{1/2}\|\rho\|_{L^\infty(\tau,+\infty;H^2(\Gamma))} \leq Q(\tau^{-1}, \mathbb{E}_0), \tag{3.36}$$

$$\|b(\rho)\|_{L^\infty(\tau,\infty;H)} \leq Q(\tau^{-1}, \mathbb{E}_0), \tag{3.37}$$

where Q is again a computable nonnegative monotone function. Moreover,

$$\|F(\rho)\|_{L^\infty(\tau,+\infty;L^{p_1}(\Omega))} \leq Q(\tau^{-1}, \mathbb{E}_0), \tag{3.38}$$

for some $p_1 > 1$. Finally, for almost all $t > 0$, the system can be rewritten in the "strong" form (1.1)–(1.4). More precisely, (1.1)–(1.2) hold as equalities in H and (1.3)–(1.4) are satisfied in H_Γ for a.e. $t > 0$.

Remark 3.8. Unfortunately, (3.33) (or (3.34)) is not satisfied by the physically relevant logarithmic potential (3.8).

Remark 3.9. In other problems characterized by singular terms and dynamic boundary conditions (cf., e.g., [8,20]), the growth assumption (3.33) (or (3.34)) can be replaced by sign conditions on g near the boundary of I (for instance, if $I = (-1, 1)$, then g has to be nonnegative near 1 and nonpositive near -1). This does not seem to be possible here, due to the term $w\mu$ in (1.4) which may be unbounded with respect to the spatial variables and "kill" the sign conditions.

Remark 3.10. The compatibility assumption (3.35) simply states that g cannot grow strictly faster than f, which is indeed true in all relevant cases. Actually, (3.35) could be strongly relaxed by paying the price of technical complications in the proof.

As a consequence of the above result, we easily obtain the existence of a *compact* absorbing set for the dynamical process $S(\cdot)$. Moreover, thanks to the contractive estimate (3.27), one can see that $S(\cdot)$ is a *closed* semigroup in the sense of Pata and Zelik [35]. Thus, by exploiting [35, Thm. 2], we immediately deduce:

Corollary 3.11. *Let the assumptions of Theorem 3.7 hold. Then, the dynamical system $S(\cdot)$ possesses a compact global attractor $\mathbb{A}$. Furthermore, there exists $c_{\mathbb{A}} \geq 0$ such that*

$$\|\rho\|_{H^{3/2}(\Omega)} + \sigma\|\rho\|_{H^2(\Omega)} + \|F(\rho)\|_{L^{p_1}(\Omega)} \leq c_{\mathbb{A}} \tag{3.39}$$

for all $\rho \in \mathbb{A}$.

Remark 3.12. Of course, (3.39) follows as a direct consequence of (3.36) and (3.38). It is also clear that it is not an optimal condition and that it could be further improved, depending on the regularity properties of f and g.

In the next subsection, we will give the proofs of the results stated so far. The analysis and characterization of ω-limit sets of single trajectories will then be presented separately in Section 3.3.

3.1. Proof of Theorem 3.2

We start by deriving several a priori estimates in order to establish the weak sequential stability of the solutions. For the sake of simplicity, we will perform these estimates by working directly, albeit formally, on the "strong" formulation (3.9)–(3.10) (or, equivalently, (3.13)–(3.14)) of the system. Actually, one should rather consider a proper approximation of system (1.1)–(1.4) (obtained, e.g., by suitably regularizing the functions f and g and the initial datum), prove that the approximate system possesses sufficiently smooth solutions, and then show that the family of these solutions satisfies the a priori estimates uniformly with respect to the approximation parameter. This has been done, in fairly complete detail, in the recent article [20], devoted to the Caginalp phase-field system with dynamic boundary conditions and singular potentials. Since the procedure required for the present system would be rather similar, we prefer to use here the formal approach and refer the reader to [20] for further details.

The energy estimate. Take $\phi = \rho_t$ in (3.9) and $\psi = \mu$ in (3.10). We then easily find

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(\rho) + \|\nabla\mu\|^2 + \delta\|\nabla_\Gamma\mu\|_\Gamma^2 = 0, \tag{3.40}$$

where the "energy" $\mathcal{E}$ was defined in (2.7). Using (fg2) to recover the full V-norm of ρ, it is then easy to infer

$$\|\rho\|_{L^\infty(0,T;V_\sigma)} \leq c, \tag{3.41}$$

$$\|F(\rho)\|_{L^\infty(0,T;L^1(\Omega))} + \|G(\rho)\|_{L^\infty(0,T;L^1(\Gamma))} \leq c, \tag{3.42}$$

$$\|\nabla\mu\|_{L^2(0,T;H)} + \delta^{1/2}\|\nabla_\Gamma\mu\|_{L^2(0,T;H_\Gamma)} \leq c. \tag{3.43}$$

Here and below, (ρ, μ) should be considered as a sequence of "approximate" solutions and the letter $c > 0$ denotes constants which are independent of the approximation parameters.

In fact, the c's can depend here on f, g and on the "magnitude" of the initial datum. We will use the letter κ to denote constants used in estimates from below, with the same dependences as c. The constants c, κ are not allowed to depend on the final time T. Specific constants will be denoted by c_i, $i \geq 1$.

A further estimate. Taking $\psi \equiv 1$ in (3.10), we observe that the dm-average $m(\rho)$ (cf. (2.4)) is conserved in time, namely,

$$m(\rho(t)) = m(\rho_0) = m_0 \quad \forall t \geq 0. \tag{3.44}$$

Then, we can take $\phi := \rho - m(\rho)$ in (3.9) and obtain

$$\langle A_\sigma \rho, \rho \rangle + (\mathcal{F}(\rho), \rho - m(\rho))_m = (\mu, \rho - m(\rho))_m, \tag{3.45}$$

where we recall that $(\cdot,\cdot)_m$ is the inner product on $\mathcal{H}$ with respect to the measure dm. Next, we observe that

$$\begin{aligned}(\mu, \rho - m(\rho))_m &= (\mu - \mu_\Omega, \rho - m(\rho))_m \\ &\leq \|\mu - \mu_\Omega\|_m \|\rho - m(\rho)\|_m \\ &\leq c\|\mu - \mu_\Omega\|_m,\end{aligned} \tag{3.46}$$

where μ_Ω is the standard spatial average of μ in Ω and, to deduce the last inequality, we have used (3.41) and the conservation property (3.44). Moreover,

$$\|\mu - \mu_\Omega\|_m \leq c\|\mu - \mu_\Omega\|_V \leq c\|\nabla \mu\|, \tag{3.47}$$

thanks to the trace theorem and the Poincaré–Wirtinger inequality. Next, proceeding as in [30, Appendix], it is not difficult to prove that

$$\begin{aligned}&(\mathcal{F}(\rho), \rho - m(\rho))_m \\ &\quad \geq \kappa \left(\int_\Omega (|f(\rho)| + F(\rho)) + \int_\Gamma (|g(\rho)| + G(\rho)) \right) - c,\end{aligned} \tag{3.48}$$

where the constant c depends on ρ only through the assigned value $m(\rho) = m_0$ (cf. (3.44)). Notice that condition (3.21) is also used here. Then, squaring (3.45) and using (3.43), we obtain

$$\|f(\rho)\|_{L^2(0,T;L^1(\Omega))} + \|g(\rho)\|_{L^2(0,T;L^1(\Gamma))} \leq c. \tag{3.49}$$

The estimate of μ. Test (3.14) using 1. Performing standard integrations and squaring, we find

$$\begin{aligned}\int_0^T \left| \int_\Omega \mu \right|^2 &\leq c \int_0^T \left| \int_\Gamma \mu \right|^2 + c\|f(\rho)\|^2_{L^2(0,T;L^1(\Omega))} \\ &\quad + c\|g(\rho)\|^2_{L^2(0,T;L^1(\Gamma))} \\ &\leq c \int_0^T \left| \int_\Gamma \mu \right|^2 + c,\end{aligned} \tag{3.50}$$

thanks to (3.49). Then, integrating in time, adding the squared $L^2(0,T;H)$-norm of $\nabla\mu$ to both sides, and using (3.43), we infer

$$\int_0^T \left| \int_\Omega \mu \right|^2 + \|\nabla \mu\|^2_{L^2(0,T;H)} \leq 2\int_0^T \left| \int_\Gamma \mu \right|^2 + c. \tag{3.51}$$

Next, noting that the quantity on the left-hand side is an equivalent norm on $L^2(0,T;V)$ and using the trace theorem and standard interpolation inequalities, we have, for $\epsilon \in (0, 1/2)$,

$$\begin{aligned}\|\mu\|^2_{L^2(0,T;V)} &\leq c(1 + \|\mu\|^2_{L^2(0,T;L^2(\Gamma))}) \\ &\leq c(1 + \|\mu\|^2_{L^2(0,T;H^{\frac12+\epsilon}(\Omega))}) \\ &\leq c + c\|\mu\|^{1+2\epsilon}_{L^2(0,T;V)} \|\mu\|^{1-2\epsilon}_{L^2(0,T;H)} \\ &\leq c + c\|\mu\|^{1+2\epsilon}_{L^2(0,T;V)} \|\mu\|^{\frac{2(1-2\epsilon)}{3}}_{L^2(0,T;V)} \|\mu\|^{\frac{(1-2\epsilon)}{3}}_{L^2(0,T;H^{-2}(\Omega))} \\ &\leq c + \frac12 \|\mu\|^2_{L^2(0,T;V)} + c\|\mu\|^2_{L^2(0,T;H^{-2}(\Omega))}.\end{aligned} \tag{3.52}$$

Now, let us take

$$\phi \in H^2_0(\Omega) \subset \mathcal{V}$$

in (3.9). Then, the Γ-components vanish, since ϕ has zero trace. Thus, using also the continuous embedding $H^2_0(\Omega) \subset L^\infty(\Omega)$, we obtain

$$|\langle \mu, \phi \rangle_{H^2_0(\Omega)}| \leq c(\|\nabla \rho\|_H + \|f(\rho)\|_{L^1(\Omega)})\|\phi\|_{H^2_0(\Omega)}. \tag{3.53}$$

Squaring, integrating over $(0,T)$, passing to the supremum with respect to ϕ with unit norm, and using (3.41) and (3.49), we then deduce that the last term on the right-hand side of (3.52) is controlled. Recalling also (3.43), we finally infer

$$\|\mu\|_{L^2(0,T;V_\delta)} \leq c. \tag{3.54}$$

Passing to the limit. In order to prove the existence of a weak solution, we exploit estimates (3.41)–(3.43), (3.49), and (3.54) to show that any sequence of approximate solutions (let us denote it by, say, (ρ_n, μ_n), where n is the approximation parameter intended to go to $+\infty$) admits at least a subsequence which converges to a weak solution of our system. Of course, this part is again formal, since the approximation has not been specified. That said, what we obtain is the existence of a pair (ρ, μ) and of a nonrelabelled subsequence of n such that

$$\rho_n \to \rho \quad \text{weak star in } L^\infty(0,T;V_\sigma), \tag{3.55}$$

$$\mu_n \to \mu \quad \text{weakly in } L^2(0,T;V_\delta). \tag{3.56}$$

In addition, using the continuity of the linear operator $\mathcal{A}_\delta$ in (3.13), we see that

$$\rho_{n,t} \to \rho_t \quad \text{weakly in } L^2(0,T;V'_\delta), \tag{3.57}$$

whence, recalling (3.55) and using the Aubin–Lions compactness lemma,

$$\rho_n \to \rho \quad \text{strongly in } C^0([0,T];H^{1-\epsilon}(\Omega)) \tag{3.58}$$

for all $\epsilon > 0$. In particular, we have the a.e. convergence of ρ_n, both in Ω and on Γ.

Next, by (3.55), (3.56), and the continuity of the linear operator $\mathcal{A}_\sigma$ in (3.14), we end up with

$$\mathcal{B}(\rho_n) \to \overline{\mathcal{B}} \quad \text{weakly in } L^2(0,T;V'_\sigma) \tag{3.59}$$

and we have to identify the limit function $\overline{\mathcal{B}}$ in some way. To do this, we first notice that (3.55)–(3.56) and (3.59) are sufficient for taking the limit of (3.14) which reads

$$\mu = \mathcal{A}_\sigma \rho + \overline{\mathcal{B}} - \lambda \mathcal{I}_{\mathrm{w}}(\rho). \tag{3.60}$$

Actually, it is immediate to take the limit of $\mathcal{I}_{\mathrm{w}}(\rho_n)$, since $\mathcal{I}_{\mathrm{w}}$ is linear and continuous. Then, we test (3.14) (written at the n-approximation level) using ρ_n in the V_σ-duality and integrate. This gives

$$\begin{aligned}\int_0^T (\mathcal{B}(\rho_n), \rho_n)_m &= \int_0^T \langle \mathcal{B}(\rho_n), \rho_n \rangle_{V_\sigma} \\ &= \int_0^T (\mu_n, \rho_n)_m - \int_0^T (\|\nabla \rho_n\|^2 + \sigma\|\nabla_\Gamma \rho_n\|^2_\Gamma) \\ &\quad + \lambda \int_0^T (\|\rho_n\|^2 + \|\rho_n\|^2_\Gamma),\end{aligned} \tag{3.61}$$

where we have used the identification of $\mathcal{H}$ with $\mathcal{H}'$ through the scalar product (2.5). Then, taking the lim sup as $n \nearrow +\infty$ of (3.61), using relations (3.55)–(3.59), and comparing with (3.60) tested by using ρ, it is not difficult to infer

$$\limsup_{n \nearrow +\infty} \int_0^T \langle \mathcal{B}(\rho_n), \rho_n \rangle_{V_\sigma} \leq \int_0^T \langle \overline{\mathcal{B}}, \rho \rangle_{V_\sigma}, \tag{3.62}$$

whence, recalling (3.18) and using, e.g., [36, Prop. 1.1, p. 42], we obtain

$$\overline{\mathcal{B}} \in \mathcal{B}_{\rm w}(\rho) \quad \text{a.e. in } (0,T) \tag{3.63}$$

and (3.60) actually coincides with (3.25). Observing that passing to the limit in (3.13) and the initial condition is immediate, thanks to (3.56)–(3.58), we have the validity of (3.24) and (3.26). Since (3.22)–(3.23) are obvious consequences of (3.55)–(3.59), this concludes the proof of existence.

Uniqueness. We will prove a contractive estimate. Let us consider two initial data $\rho_{0,1}$, $\rho_{0,2}$ both belonging to the space $\mathcal{X}_0$ (and, hence, having in particular the same dm-average m_0). Let (ρ_1,μ_1), (ρ_2,μ_2) be the solutions corresponding to these initial data and let us denote by (ρ,μ) their difference. We take the difference of (3.24) (written for both solutions), integrate with respect to time, and test the resulting equation with μ in the V_δ-duality. This gives

$$(\rho,\mu)_m = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega |\nabla(1*\mu)|^2 + \delta\int_\Gamma |\nabla_\Gamma(1*\mu)|^2 - 2(\rho_0, 1*\mu)_m\right) \tag{3.64}$$

and, using the fact that $m(\rho_0) = 0$ and the Poincaré–Wirtinger inequality, we obtain

$$\begin{aligned} 2|(\rho_0,1*\mu)_m| &\le 2\|\rho_0\|_{V_\delta'}\|1*\mu-(1*\mu)_\Omega\|_{V_\delta} \\ &\le c\|\rho_0\|^2_{V_\delta'} + \frac{1}{2}\|\nabla(1*\mu)\|^2 \\ &\quad + \frac{\delta}{2}\|\nabla_\Gamma(1*\mu)\|^2_\Gamma. \end{aligned} \tag{3.65}$$

Next, testing the difference of (3.25) (written, again, for both solutions) with ρ in the V_σ-duality and using the monotonicity of $\mathcal{B}_{\rm w}$, we infer

$$(\rho,\mu)_m \ge \|\nabla\rho\|^2 + \sigma\|\nabla_\Gamma\rho\|^2_\Gamma - \lambda(\|\rho\|^2 + \|\rho\|^2_\Gamma) \tag{3.66}$$

and we have to control the last term on the right-hand side. To do this, we first notice that, by standard trace theorems, for $\epsilon \in (0,1/2)$, it holds that

$$\lambda(\|\rho\|^2+\|\rho\|^2_\Gamma) \le c\|\rho\|^2_{H^{\frac{1}{2}+\epsilon}(\Omega)} \le \frac{1}{2}\|\nabla\rho\|^2 + c\|\rho\|^2_{H^{-1}(\Omega)}, \tag{3.67}$$

the latter inequality following from the chain of compact embeddings $V \subset H^{1/2+\epsilon}(\Omega) \subset H^{-1}(\Omega)$. Thus, collecting (3.64)–(3.67), we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega |\nabla(1*\mu)|^2 + \delta\int_\Gamma |\nabla_\Gamma(1*\mu)|^2 - 2(\rho_0,1*\mu)_m\right) + \frac{1}{2}\|\nabla\rho\|^2 + \sigma\|\nabla_\Gamma\rho\|^2_\Gamma \le c_1\|\rho\|^2_{H^{-1}(\Omega)}. \tag{3.68}$$

We now consider the linear operator $B : H^2(\Omega)\cap H^1_0(\Omega) \to H$ defined by

$$Bu = h \iff -\Delta u = h \quad \text{and} \quad u|_\Gamma = 0. \tag{3.69}$$

Then, we integrate the difference of (3.24) (written for both solutions) in time and test the resulting equality with $KB^{-1}\rho$, for $K>0$ to be chosen, to obtain

$$\begin{aligned} K\|\rho\|^2_{H^{-1}(\Omega)} &\le K\left|\int_\Omega \nabla(1*\mu)\cdot\nabla B^{-1}\rho\right| + K(\rho_0, B^{-1}\rho)_m \\ &\le \frac{K}{2}\|\rho\|^2_{H^{-1}(\Omega)} + cK\|\nabla(1*\mu)\|^2 \\ &\quad + cK\|\rho_0\|^2_{H^{-1}(\Omega)}. \end{aligned} \tag{3.70}$$

Thus, summing (3.68) and (3.70), choosing $K \ge 4c_1$, applying Gronwall's lemma, and recalling (3.65), we finally obtain the contractive estimate

$$\begin{aligned} \|\rho\|^2_{L^2(0,T;V_\sigma)} &\le c(\|\rho\|^2_{L^2(0,T;H^{-1}(\Omega))} + \|\nabla\rho\|^2_{L^2(0,T;H)} \\ &\quad + \sigma\|\nabla_\Gamma\rho\|^2_{L^2(0,T;H_\Gamma)}) \\ &\le c_T(\|\rho_0\|^2_{H^{-1}(\Omega)} + \|\rho_0\|^2_{V_\delta'}), \end{aligned} \tag{3.71}$$

whence (3.27) follows by observing that $\|\cdot\|_{H^{-1}(\Omega)} \le c\|\cdot\|_{V_\delta'}$. In particular, in the case when $\rho_{0,1} = \rho_{0,2}$, then $\rho_1 = \rho_2$ almost everywhere, i.e., we have the uniqueness of ρ. On the other hand, our argument only implies that $\nabla(1*\mu) = 0$ almost everywhere, whence μ_1 differs from μ_2 by a function which is constant in space, but may be time dependent. However, the possible multivalued character of $\mathcal{F}_{\rm w}$ prevents us from using Eq. (3.25) to prove that such a constant vanishes. □

3.2. Regularization properties of the solutions

Proof of Theorem 3.5. The procedure is standard. It is however worth noting that it is still necessary to proceed by working on the approximate statement and then passing to the limit. The reason for this is that we need to consider the operator $\mathcal{F}$ in its "strong" form in order to use pointwise arguments. That said, it is sufficient to combine the "energy estimate" and the "further estimate". More precisely, we sum (3.40) and η times (3.45), where $\eta \in (0,1)$ will be chosen later. Then, proceeding as in (3.46)–(3.48), it is not difficult to find

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(\rho) + \|\nabla\mu\|^2 + \delta\|\nabla_\Gamma\mu\|^2_\Gamma + \eta\|\nabla\rho\|^2 + \sigma\eta\|\nabla_\Gamma\rho\|^2_\Gamma \\ &\quad + \kappa\eta\left(\int_\Omega(|f(\rho)|+F(\rho)) + \int_\Gamma(|g(\rho)|+G(\rho))\right) \\ &\le c\eta\|\nabla\mu\|\,\|\rho-m(\rho)\|_m + c\eta \end{aligned} \tag{3.72}$$

for some $\kappa > 0$. From this point on, the constants κ, c, and c_i are allowed to depend on the initial datum through $m_0 = m(\rho_0)$ only. Then, the right-hand side of (3.72) can be estimated as follows:

$$\begin{aligned} &\le c\eta\|\nabla\mu\|(\|\rho\|_V + c) + c\eta \\ &\le c_2\eta^{1/2}\|\nabla\mu\|^2 + c_2\eta^{3/2}(\|\rho\|^2 + \|\nabla\rho\|^2) + c. \end{aligned} \tag{3.73}$$

Now, thanks to assumption (fg2), F is superquadratic and, consequently, there exists c_3 depending on c_2 and κ, but not on η, such that

$$\frac{\kappa\eta}{2}\int_\Omega F(\rho) - c_2\eta^{3/2}\|\rho\|^2 \ge -c_3 \tag{3.74}$$

for all $\eta \in (0,1)$. Thus, choosing η small enough that $c_2\eta^{1/2} \le 1/2$, we finally end up with

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(\rho) + \frac{1}{2}\|\nabla\mu\|^2 + \delta\|\nabla_\Gamma\mu\|^2_\Gamma + \frac{\eta}{2}\|\nabla\rho\|^2 + \sigma\eta\|\nabla_\Gamma\rho\|^2_\Gamma \\ &\quad + \frac{\kappa}{2}\eta\left(\int_\Omega(|f(\rho)|+F(\rho)) + \int_\Gamma(|g(\rho)|+G(\rho))\right) \le c. \end{aligned} \tag{3.75}$$

Recalling (2.7), we obtain, in particular,

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(\rho) + \kappa\mathcal{E}(\rho) + \kappa\|\nabla\mu\|^2 + \kappa\delta\|\nabla_\Gamma\mu\|^2_\Gamma \\ &\quad + \kappa\left(\int_\Omega |f(\rho)| + \int_\Gamma |g(\rho)|\right) \le c \end{aligned} \tag{3.76}$$

for some (new) value of $\kappa > 0$. Then, Gronwall's lemma immediately gives the *dissipative estimate*

$$\mathcal{E}(\rho(t)) + \kappa \int_t^{t+1} \left(\|\nabla\mu\|^2 + \delta\|\nabla_\Gamma \mu\|_\Gamma^2 + \int_\Omega |f(\rho)| + \int_\Gamma |g(\rho)| \right) \le \mathbb{E}_0 e^{-\kappa t} + c, \quad \forall t \ge 0, \tag{3.77}$$

whence the thesis of the theorem. □

Proof of Theorem 3.6. The main ingredient is a new a priori estimate (as above, we derive it formally, although we should still work on some kind of approximation). We compute the time derivative of (3.9),

$$\int_\Omega \mu_t \phi + \int_\Gamma \mu_t \phi w = \int_\Omega \nabla\rho_t \cdot \nabla\phi + \int_\Omega f'(\rho)\rho_t\phi + \sigma \int_\Gamma \nabla_\Gamma \rho_t \cdot \nabla_\Gamma \phi + \int_\Gamma g'(\rho)\rho_t\phi. \tag{3.78}$$

Then, we take $\phi = \rho_t$ in the above formula and also set $\psi = \mu_t$ in (3.10). This clearly gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|\nabla\mu\|^2 + \frac{\delta}{2}\|\nabla_\Gamma \mu\|_\Gamma^2\right) + \|\nabla\rho_t\|^2 + \sigma\|\nabla_\Gamma \rho_t\|_\Gamma^2 = -\int_\Omega f'(\rho)\rho_t^2 - \int_\Gamma g'(\rho)\rho_t^2 \le \lambda(\|\rho_t\|^2 + \|\rho_t\|_\Gamma^2). \tag{3.79}$$

To estimate the right-hand side, we proceed as in the proof of uniqueness. More precisely, we first observe that, at least at the approximate level, $\rho_t \in H$, say, a.e. in time. Thus, we can test (3.13) with $B^{-1}\rho_t$ (cf. (3.69)). As in (3.70), we obtain

$$\|\rho_t\|_{H^{-1}(\Omega)}^2 = -\int_\Omega \nabla\mu \cdot \nabla B^{-1}\rho_t \le \|\nabla\mu\| \, \|\nabla B^{-1}\rho_t\| \le \frac{1}{2}\|\rho_t\|_{H^{-1}(\Omega)}^2 + c\|\nabla\mu\|^2. \tag{3.80}$$

Summing now (3.79) and K times (3.80), with $K > 0$ to be chosen, and noting that (cf. (3.67))

$$\lambda\|\rho_t\|^2 + \lambda\|\rho_t\|_\Gamma^2 \le \frac{1}{2}\|\nabla\rho_t\|^2 + c_4\|\rho_t\|_{H^{-1}(\Omega)}^2, \tag{3.81}$$

we can take $K \ge 4c_4$. Then, the *uniform* Gronwall lemma (see, e.g., [37, Lemma 1.1.1]), also on account of the dissipative estimate (3.77), gives (3.31). Moreover, we obtain a locally uniform bound on the gradient of μ. To have the full V_δ-norm of μ, we first have to come back to the "further estimate" and notice that, thanks to the above procedure, (3.49) can now be improved (and rephrased) as

$$\|f(\rho)\|_{L^\infty(\tau,+\infty;L^1(\Omega))} + \|g(\rho)\|_{L^\infty(\tau,+\infty;L^1(\Gamma))} \le Q(\tau^{-1}, \mathbb{E}_0). \tag{3.82}$$

Then, we go back to the "estimate of μ" and repeat (3.50)–(3.53), but *without* integrating in time. We then obtain, pointwise in time,

$$\|\mu\|_V \le c\|f(\rho)\|_{L^1(\Omega)} + c\|g(\rho)\|_{L^1(\Gamma)} + c(\|\nabla\rho\|_H + \|f(\rho)\|_{L^1(\Omega)}). \tag{3.83}$$

Finally, for any $t \ge \tau > 0$, we can compute the essential supremum of the right-hand side of (3.83) over $(t, t+1)$. By (3.77) and (3.82), this gives (3.32) and concludes the proof. □

Proof of Theorem 3.7. Let us now fix $\tau > 0$ and take $t \ge \tau$ so that properties (3.31)–(3.32) hold. We then consider, for $t \ge \tau$, the elliptic system

$$-\Delta\rho + b(\rho) = \mu + \lambda\rho, \tag{3.84}$$

$$-\sigma\Delta_\Gamma \rho + g(\rho) = w\mu - \partial_{\mathbf{n}}\rho, \tag{3.85}$$

where we have set $b(r) := f(r) + \lambda r$, for $r \in I$. Notice that, in the framework of weak solutions, we are not allowed to intend (3.84)–(3.85) in the present (strong) form. However, this may be done at the approximate level. For simplicity, let us proceed formally and look for (pointwise in time) estimates of the nonlinear terms. In what follows, we denote by C a constant behaving as the right-hand side of (3.31)–(3.32), i.e., monotonically depending on $\mathbb{E}_0$ and τ^{-1}.

Then, we test both (3.84) and (3.85) by $b(\rho)$ and integrate by parts. We obtain

$$\|b(\rho)\|^2 + \int_\Gamma g(\rho)b(\rho) + \int_\Omega b'(\rho)|\nabla\rho|^2 + \sigma\int_\Gamma b'(\rho)|\nabla_\Gamma\rho|^2 \le \int_\Omega (\mu + \lambda\rho)b(\rho) + \int_\Gamma w\mu b(\rho). \tag{3.86}$$

Let us first provide an estimate for the right-hand side. We first note that

$$\int_\Omega (\mu + \lambda\rho)b(\rho) \le \frac{1}{4}\|b(\rho)\|^2 + c(\|\mu\|^2 + \|\rho\|^2) \le \frac{1}{4}\|b(\rho)\|^2 + C, \tag{3.87}$$

thanks to (3.77) and (3.32). Next, we observe that

$$\int_\Gamma w\mu b(\rho) \le \|w\mu\|_{L^p(\Gamma)}\|b(\rho)\|_{L^q(\Gamma)}, \tag{3.88}$$

where p and q are conjugate exponents with $p > 4/3$. Using standard trace theorems, we find

$$\|b(\rho)\|_{L^q(\Gamma)} \le c\|b(\rho)\|_{W^{1,\frac{3q}{2+q}}(\Omega)} \le c(\|b(\rho)\| + \|b'(\rho)\nabla\rho\|_{L^{\frac{3q}{2+q}}(\Omega)}). \tag{3.89}$$

Next, we notice that

$$\|b'(\rho)\nabla\rho\|_{L^{\frac{3q}{2+q}}(\Omega)} \le \|b'(\rho)^{1/2}\nabla\rho\| \, \|b'(\rho)^{1/2}\|_{L^{\frac{6q}{4-q}}(\Omega)} \tag{3.90}$$

and, of course, we also have

$$\|w\mu\|_{L^p(\Gamma)}\|b(\rho)\| \le \frac{1}{4}\|b(\rho)\|^2 + \|w\mu\|_{L^p(\Gamma)}^2. \tag{3.91}$$

Thus, collecting (3.88)–(3.91), we end up with

$$\int_\Gamma w\mu b(\rho) \le \frac{1}{2}\|b'(\rho)^{1/2}\nabla\rho\|^2 + \frac{1}{4}\|b(\rho)\|^2 + c\|w\mu\|_{L^p(\Gamma)}^2(\|b'(\rho)\|_{L^{\frac{3q}{4-q}}(\Omega)} + 1) \tag{3.92}$$

and, expressing q in terms of p, we finally have

$$\frac{1}{2}\|b(\rho)\|^2 + \int_\Gamma g(\rho)b(\rho) + \frac{1}{2}\int_\Omega b'(\rho)|\nabla\rho|^2 + \sigma\int_\Gamma b'(\rho)|\nabla_\Gamma\rho|^2 \le c\|w\mu\|_{L^p(\Gamma)}^2\left(\|b'(\rho)\|_{L^{\frac{3p}{3p-4}}(\Omega)} + 1\right) + C \le c_\eta\|w\mu\|_{L^p(\Gamma)}^{\frac{3p}{2}} + \eta\|b'(\rho)\|_{L^{\frac{3p}{3p-4}}(\Omega)}^{\frac{3p}{3p-4}} + C, \tag{3.93}$$

where $\eta > 0$ is a (small) constant to be chosen. Now, the procedure differs, depending on the value of δ. Indeed, if $\delta > 0$, we obtain, from (2.2) and (3.32),

$$\|w\mu\|_{L^p(\Gamma)} \le C_p \quad \forall p \in (4/3, +\infty), \tag{3.94}$$

the constant C_p depending on p. Thus, by (3.33) (which holds, of course, also for b, since it is a linear perturbation of f), we can take p large enough that

$$\eta \int_\Omega |b'(\rho)|^{\frac{3p}{3p-4}} \le c_f \eta \int_\Omega |b(\rho)|^{\frac{3pp_0}{3p-4}} + c \le \frac{1}{4}\int_\Omega |b(\rho)|^2 + c \tag{3.95}$$

and the right-hand side is controlled by the first term in (3.93). Instead, if $\delta = 0$, we just have

$$\|w\mu\|_{L^4(\Gamma)} \le c, \tag{3.96}$$

thanks to the continuity of the trace operator from V to $L^4(\Gamma)$. Thus, we take $p = 4$ and obtain, using (3.34) (which still holds for b),

$$\eta \int_\Omega |b'(\rho)|^{\frac{3}{2}} \le c_f \eta \int_\Omega |b(\rho)|^2 + c \le \frac{1}{4}\int_\Omega |b(\rho)|^2 + c, \tag{3.97}$$

provided that we choose η small enough. Therefore, in both cases, we can control the right-hand side of (3.86) (or, equivalently, of (3.93)).

To obtain an estimate from (3.93), we still have to control the second term on the left-hand side. Actually, if f is a "regular" function (i.e., $I = \mathbb{R}$), the monotonicity of b and the second of (fg2) show that b and g have the same sign for $|r|$ large enough. Thus, clearly, thanks also to (3.35),

$$\int_\Gamma g(\rho)b(\rho) \ge \kappa \|g(\rho)\|_\Gamma^2 - c. \tag{3.98}$$

If $I \ne \mathbb{R}$, we have to estimate this term directly. Noting that, by (fg1), g is uniformly bounded on $\bar{I}$ and using Gagliardo's trace theorem (cf., e.g., [38]), we obtain

$$\begin{aligned}\left|\int_\Gamma g(\rho)b(\rho)\right| &\le c\|b(\rho)\|_{L^1(\Gamma)} \le c\|b(\rho)\|_{W^{1,1}(\Omega)} \\ &\le c\|b(\rho)\|_{L^1(\Omega)} + c\|b'(\rho)\nabla\rho\|_{L^1(\Omega)} \\ &\le c\|b(\rho)\|_{L^1(\Omega)} + c_\eta\|b'(\rho)\|_{L^1(\Omega)} \\ &\quad + \eta\|b'(\rho)|\nabla\rho|^2\|_{L^1(\Omega)}\end{aligned} \tag{3.99}$$

and the last term on the right-hand side is estimated, for η small enough, by the third term on the left-hand side of (3.93), while the first two terms, as above, can be controlled by the first term in (3.93), thanks to (3.33) or (3.34) (we leave the details to the reader).

Since the above estimates hold pointwise in time, passing to the supremum with respect to $t \in [\tau, +\infty)$ and recalling the "meaning" of the constant C, we have

$$\|b(\rho)\|_{L^\infty(\tau,+\infty;H)} \le Q(\mathbb{E}_0, \tau^{-1}). \tag{3.100}$$

Moreover, thanks to the boundedness of ρ (for "singular" f) or to (3.98) (for "regular" f), we also have

$$\|g(\rho)\|_{L^\infty(\tau,+\infty;H_\Gamma)} \le Q(\mathbb{E}_0, \tau^{-1}). \tag{3.101}$$

In particular, with (3.100)–(3.101) at our disposal, at least for $t \ge \tau > 0$ we can identify, the nonlinear terms with respect to the topology of $\mathcal{H}$ (rather than in the duality between V'_σ and V_σ as we did in (3.59)–(3.63)). In other words, (3.37) holds and we can replace $\mathcal{F}_w(\rho)$ with $\mathcal{F}(\rho)$ in (3.25). Hence, (3.24)–(3.25) can be rewritten as (3.13)–(3.14) or, equivalently, (3.9)–(3.10).

In particular, (3.9) can be seen as a weak formulation of (3.84)–(3.85). Then, we can apply standard elliptic regularity results (see, e.g., [20, Lemma 2.2 and Rem. 2.3] for a precise statement) to obtain an H^2-estimate for ρ if $\sigma > 0$ and an $H^{3/2}$-estimate if $\sigma = 0$. Namely, we have proved (3.36). At this point, if we integrate by parts the gradient terms in (3.9)–(3.10), estimates (3.31)–(3.32), (3.100)–(3.101), and (3.36) allow us to prove, by comparison arguments, that

$$\|\Delta\rho\|_{L^\infty(\tau,+\infty;H)} + \sup_{t\ge\tau}\|\Delta\mu\|_{L^2(t,t+1;V)} \le Q(\mathbb{E}_0, \tau^{-1}) \tag{3.102}$$

and to give sense to the normal derivatives. More precisely, by the first of (3.102), (3.36), and proper trace theorems, we have

$$\begin{aligned}&\|\partial_{\boldsymbol{n}}\rho\|_{L^\infty(\tau,+\infty;H^{-1/2}(\Gamma))} + \sigma^{1/2}\|\partial_{\boldsymbol{n}}\rho\|_{L^\infty(\tau,+\infty;H^{1/2}(\Gamma))} \\ &\quad \le Q(\mathbb{E}_0, \tau^{-1}).\end{aligned} \tag{3.103}$$

From this relation, we can see that, for all $t > 0$, (1.4) makes sense as an equality in H_Γ (actually, in the case $\sigma = 0$, we first obtain it in the space $H^{-1/2}(\Gamma)$. Then, we see that it holds in H_Γ by simply comparing terms). Analogously, the second of (3.102) gives

$$\sup_{t\ge\tau}\|\partial_{\boldsymbol{n}}\mu\|_{L^2(t,t+1;H^{-1/2}(\Gamma))} \le Q(\mathbb{E}_0, \tau^{-1}), \tag{3.104}$$

so (1.3) can be interpreted at least in $H^{-1/2}(\Gamma)$. Actually, we can say more. Indeed, in the case $\delta = 0$, $\partial_{\boldsymbol{n}}\mu$ equals $-w\rho_t$; thus, (1.3) holds in H_Γ, by (3.36). The same holds for $\delta > 0$, since we can view (1.1) + (1.3) as a coupled (time-dependent) elliptic system and apply once more [20, Lemma 2.2]. Hence, we have, in all cases,

$$\sup_{t\ge\tau}\|\partial_{\boldsymbol{n}}\mu\|_{L^2(t,t+1;H_\Gamma)} \le Q(\mathbb{E}_0, \tau^{-1}) \tag{3.105}$$

and, summarizing, (3.9)–(3.10) can be interpreted in the strong form (1.1)–(1.4).

To conclude the proof, we have to show (3.38), which is, however, immediate. Actually, in the singular case, it is clear that $|F(r)| \le c(1+|b(r)|)$ for any $r \in I$, whence (3.38) holds for $p_1 = 2$, thanks to (3.100). In the regular case, i.e., if $I = \mathbb{R}$, using the monotonicity of b, we have, instead,

$$F(r) = \int_0^r (b(s) - \lambda s)\mathrm{d}s \le |rb(r)| + cr^2, \tag{3.106}$$

whence (3.38) now follows from (3.100), (3.36), and interpolation (again, for $p_1 = 2$ if $\sigma > 0$ and for all $p_1 \in (1, 2)$ if $\sigma = 0$). The proof of the theorem is complete. □

Remark 3.13. If the initial datum is more regular, one can use the standard (instead of the uniform) Gronwall lemma in the proof of Theorem 3.6 and see that estimates (3.31)–(3.32) (as well as all the subsequent ones) also hold for $\tau = 0$, with the right-hand sides replaced by a quantity suitably depending on some better norm of the initial datum. To be more precise, what is needed is that the term under the time derivative on the left-hand side of (3.79) is controlled at $t = 0$, namely,

$$\mu|_{t=0} \in V_\delta. \tag{3.107}$$

Notice, however, that $\mu|_{t=0}$ is not prescribed as an initial datum and, in fact, (3.107) has to be obtained by asking for more regularity on ρ_0 (both in Ω and on Γ) and deducing the corresponding regularity on $\mu|_{t=0}$ by comparison in (1.2) and in (1.4) evaluated at $t = 0$. We leave the details to the reader.

Remark 3.14. The growth condition (3.33) or (3.34) could actually be relaxed a bit. Indeed, in the above procedure, one could test the equations with $|b(\rho)|^s$ sign $(b(\rho))$ for a suitable $s \in (0, 1)$ in order to look for an estimate of $b(\rho)$ in $L^{s+1}(\Omega)$, rather than in H. Such a weaker estimate is compatible with a (slightly) more general class of nonlinearities; it is anyway still sufficient, of course, to interpret the system in the stronger form. We do not give the details, since the procedure would involve a number of boring technical complications to provide just a small extension of the result.

3.3. Characterization of ω-limit sets

In this final subsection, we deal with the long-time behavior of single trajectories and use the well-posedness and regularization results to characterize their ω-limit sets. We will always assume that the hypotheses of Theorem 3.7 (among which, in particular, (3.33)–(3.35)) hold, so that the evolutionary system can be interpreted in the strong form (1.1)–(1.4), at least for $t > 0$.

Our first step consists in analyzing the stationary problem associated with (1.1)–(1.4). As far as the μ-component is concerned, it reads

$$-\Delta\mu_\infty = 0, \quad \text{in } \Omega, \tag{3.108}$$

$$-\delta\Delta_\Gamma\mu_\infty = -\partial_{\boldsymbol{n}}\mu_\infty, \quad \text{on } \Gamma \tag{3.109}$$

and, clearly, μ_∞ solves the system if and only if it is a constant function. Thus, the ρ-system becomes

$$-\Delta\rho_\infty + f(\rho_\infty) = \mu_\infty, \quad \text{in } \Omega, \tag{3.110}$$

$$-\sigma\Delta_\Gamma\rho_\infty + g(\rho_\infty) = w\mu_\infty - \partial_{\boldsymbol{n}}\rho_\infty, \quad \text{on } \Gamma, \tag{3.111}$$

where μ_∞ is a constant which is, at least a priori, undetermined.

We can now prove the existence of (nonempty) ω-limit sets of weak solutions.

Theorem 3.15. *Let the assumptions of Theorem 3.7 hold and let (ρ, μ) be a solution. Then, the function $t \mapsto \rho(t)$ possesses a nonempty ω-limit set which only consists of solutions ρ_∞ of the stationary problem (3.110)–(3.111). More precisely, for any sequence $\{t_n\}$ diverging to $+\infty$, there exist a nonrelabelled subsequence and a solution $(\rho_\infty, \mu_\infty)$ of (3.108)–(3.111) such that*

$$\rho(t_n) \to \rho_\infty, \qquad \mu(t_n) \to \mu_\infty \tag{3.112}$$

in suitable topologies. Moreover, any such ρ_∞ satisfies, in addition,

$$m(\rho_\infty) = m_0, \tag{3.113}$$

where m_0 is the dm*-average of the initial datum ρ_0.*

Proof. Let $\{t_n\} \subset [0, +\infty)$ be any sequence of times diverging to $+\infty$. Then, thanks to (3.36), we see that, up to a (nonrelabelled) subsequence, $\rho(t_n)$ tends to some limit ρ_∞ strongly in V_σ. We can set, for $t \in [0, 1]$, $(\rho_n, \mu_n)(t) := (\rho, \mu)(t_n + t)$ and notice that (ρ_n, μ_n) solves, for $t \in [0, 1]$, problem (1.1)–(1.4). Moreover, it satisfies the "initial" condition $\rho_n|_{t=0} = \rho(t_n)$. Furthermore, due to (3.36), it holds that

$$\rho_n \to \widehat{\rho} \quad \text{weakly star in } L^\infty(0, 1; V_\sigma), \tag{3.114}$$

while (3.32) entails

$$\mu_n \to \widehat{\mu} \quad \text{weakly star in } L^\infty(0, 1; V_\delta). \tag{3.115}$$

Additionally, integrating (3.40) over $(0, +\infty)$, we find

$$\int_0^{+\infty} (\|\nabla\mu\|^2 + \delta\|\nabla_\Gamma\mu\|_\Gamma^2) \le \mathbb{E}_0, \tag{3.116}$$

which, using the continuity of the linear operator $\mathcal{A}_\delta$ in (3.24), also entails

$$\int_0^{+\infty} \|\rho_t\|_{V_\delta'}^2 \le Q(\mathbb{E}_0). \tag{3.117}$$

Consequently, we have

$$\rho_{n,t} \to 0 \quad \text{strongly in } L^2(0, 1; V_\delta'), \tag{3.118}$$

which yields that $\widehat{\rho}$ in (3.114) is constant in time and, hence, coincides with $\rho_\infty = \lim \rho_n(0)$.

At this point, we can take the limit of (1.1)–(1.4), written for (ρ_n, μ_n), over the time interval $(0, 1)$. To do this, we first notice that, by (3.114), (3.100)–(3.101),

$$b(\rho_n) \to \overline{b} \quad \text{weakly star in } L^\infty(0, 1; H), \tag{3.119}$$

$$g(\rho_n) \to \overline{g} \quad \text{weakly star in } L^\infty(0, 1; H_\Gamma). \tag{3.120}$$

Moreover, by (3.114), (3.118), and the Aubin–Lions lemma,

$$\rho_n \to \rho_\infty \quad \text{strongly in } L^2(0, 1; \mathcal{H}). \tag{3.121}$$

Thanks to the standard monotonicity argument [36, Prop. 1.1, p. 42], we then have

$$\overline{b} = b(\rho_\infty), \quad \text{a.e. in } (0, 1) \times \Omega \tag{3.122}$$

and, analogously, $\overline{g}$ coincides a.e. on Γ with $g(\rho_\infty)$.

Summarizing, we see that ρ_∞ and $\widehat{\mu}$ solve, a.e. in the time interval $(0, 1)$, the system

$$-\Delta\widehat{\mu} = 0, \quad \text{in } \Omega, \tag{3.123}$$

$$-\delta\Delta_\Gamma\widehat{\mu} = -\partial_{\boldsymbol{n}}\widehat{\mu}, \quad \text{on } \Gamma, \tag{3.124}$$

$$-\Delta\rho_\infty + f(\rho_\infty) = \widehat{\mu}, \quad \text{in } \Omega, \tag{3.125}$$

$$-\sigma\Delta_\Gamma\rho_\infty + g(\rho_\infty) = w\widehat{\mu} - \partial_{\boldsymbol{n}}\rho_\infty, \quad \text{on } \Gamma. \tag{3.126}$$

From (3.125)–(3.126), it is then clear that also $\widehat{\mu}$ is constant with respect to time. Thus, we can denote it by μ_∞. Moreover, (3.123)–(3.124) imply that μ_∞ is also constant with respect to the space variables. Then, (3.124)–(3.125) actually reduces to (3.109)–(3.110). Finally, since $m(\rho(t_n)) = m_0$ for all n, thanks to (3.44), (3.113) immediately follows by letting $n \nearrow +\infty$, which concludes the proof. $\square$

As a next step, we analyze a bit more carefully the set of the solutions of the stationary problem (3.110)–(3.111) which can be reached in the ω-limit. We first have:

Lemma 3.16. *Let the assumptions of Theorem 3.7 hold and let (ρ, μ) be a solution. Then, there exists a constant c_5 depending on the initial datum only through m_0 such that, for any μ_∞ in the ω-limit set of ρ, it holds that*

$$|\mu_\infty| \le c_5. \tag{3.127}$$

Proof. We test both (3.110) and (3.111) with $\rho_\infty - m_0$, where ρ_∞ lies in the ω-limit set of ρ (and, hence, satisfies, in addition, (3.113)). Proceeding as in (3.48), we then have

$$\begin{aligned} &\|\nabla\rho_\infty\|^2 + \sigma\|\nabla_\Gamma\rho_\infty\|_\Gamma^2 \\ &\quad + \kappa\left(\int_\Omega (|f(\rho_\infty)| + F(\rho_\infty)) + \int_\Gamma (|g(\rho_\infty)| + G(\rho_\infty))\right) \\ &\le (\mu_\infty, \rho_\infty - m_0)_m + c = c. \end{aligned} \tag{3.128}$$

Indeed, the latter scalar product vanishes, since μ_∞ is a constant and (3.113) holds. Consequently, we have

$$\|\rho_\infty\|_{V_\sigma}^2 + \|f(\rho_\infty)\|_{L^1(\Omega)} + \|g(\rho_\infty)\|_{L^1(\Gamma)} \le c. \tag{3.129}$$

The thesis then follows by integrating both (3.110) and (3.111) with respect to the space variables and using (3.129). $\square$

Our next aim is to show that there exists a compact subinterval $I_1 \subset I$ such that any element ρ_∞ of the ω-limit set can only take values in I_1. In other words, ρ_∞ is bounded in the regular case and is "separated from the singularities" in the singular one. However, in the singular case this requires some additional hypotheses.

Lemma 3.17. *Let the assumptions of Theorem 3.7 hold and let (ρ, μ) be a solution. Let us also assume that either f is regular (i.e., $I = \mathbb{R}$)*

or there exists $\varepsilon > 0$ such that

$$g(r) - w^* c_5 \geq 0 \quad \forall r \geq r^* - \varepsilon, \\ g(r) + w^* c_5 \leq 0 \quad \forall r \leq r_* + \varepsilon, \tag{3.130}$$

where c_5 is the constant in (3.127) and we have written $I = (r_*, r^*)$. Then, there exists a compact subinterval $I_1 \subset I$ such that any element ρ_∞ of the ω-limit set of ρ takes values in I_1.

Proof. We test (3.110) with $(\rho_\infty - M)^+$ for $M > 0$ to be chosen. Using (3.111), we then have

$$\|\nabla(\rho_\infty - M)^+\|^2 + \sigma \|\nabla_\Gamma(\rho_\infty - M)^+\|_\Gamma^2 \\ + \int_\Omega (f(\rho_\infty) - \mu_\infty)(\rho_\infty - M)^+ \\ + \int_\Gamma (g(\rho_\infty) - w\mu_\infty)(\rho_\infty - M)^+ = 0. \tag{3.131}$$

It is now clear that, if f is regular, then, thanks to (2.2), (fg2) and (3.127), we can always choose M large enough that the last two terms in the left-hand side are nonnegative. On the other hand, if f is singular, then M has to stay in I. Thus, nothing changes in what relates to the third term in (3.131) which can still be made nonnegative by taking M close enough to r^*. Concerning, instead, the boundary term, in order to ensure the same, we need the first of (3.130). The lower bound is then obtained in a similar way by testing (3.110) with $-(\rho_\infty + M)^-$ and using the second inequality of (3.130). □

Remark 3.18. Since the constant c_5 depends, in particular, on m_0, (3.130) can be seen as an assumption of compatibility between g, m_0, and w which is more likely to be satisfied when m_0 is close to 0. Of course, it may very well happen that there exists some nonlinearity g satisfying all other hypotheses, but for which (3.130) holds for no value of m_0. In such a case, the elements of the ω-limit set may not be separated from the singularities of f and the subsequent analysis cannot be performed.

We now show that, under the same hypotheses on the nonlinear terms, also the solutions of the evolution problem are separated from the singularities of f, at least for sufficiently large times.

Lemma 3.19. *Let the assumptions of Lemma 3.17 (among which, in particular, is (3.130), in the case of a singular function f) hold and let (ρ, μ) be a solution. Then, there exist a compact subinterval $I_2 \subset I$ and a time T_1 depending on the solution such that $\rho(t, x) \in I_2$ for almost all $x \in \Omega$ and $t \geq T_1$.*

Proof. We only give the proof in the case $\sigma > 0$, which is almost immediate. Actually, if f is regular, i.e., $I = \mathbb{R}$, then the result directly follows from (3.36) and the continuous embedding

$$H^2(\Omega) \subset C^{0,\alpha}(\overline{\Omega}), \quad \alpha \in (0, 1/2). \tag{3.132}$$

In the case of a singular function f, the assertion follows from the uniform separation of any element of the ω-limit set (cf. Lemma 3.17), from relation (3.36) again, and from the embedding (3.132) which entails that the elements of the ω-limit set are reached with respect to the uniform convergence. So, one can take any compact subinterval I_2 of I such that I_1 is contained in the interior of I_2.

As regards the case $\sigma = 0$, the same argument does not apply, since, from (3.36), we can only obtain precompactness in the $H^{3/2}$-norm, which does not entail uniform convergence. Nonetheless, this problem can be overcome by proving a further estimate on $f(\rho)$, whose details can be found in [9, Lemma 6.3]. □

Lemma 3.20. *Let the assumptions of Lemma 3.17 hold and let (ρ, μ) be a solution. Additionally, assume that*

$$f \in C^{1,1}_{\text{loc}}(I, \mathbb{R}), \qquad g \in C^{1,1}_{\text{loc}}(\mathbb{R}, \mathbb{R}), \qquad w \in W^{1,\infty}(\Gamma). \tag{3.133}$$

Then, there exist a time T_2 and a constant C depending on the solution such that

$$\|\rho\|_{L^\infty(T_2,+\infty;H^{5/2}(\Omega))} + \sigma^{1/2} \|\rho\|_{L^\infty(T_2,+\infty;H^3(\Omega))} \leq C. \tag{3.134}$$

Proof. We take, for $t \geq T_1$, the time derivatives of (3.9) and of (3.10) and set $\phi = \rho_{tt}$ and $\psi = \mu_t$ therein. Using the previous lemma and the local Lipschitz continuity of f' and g' given by (3.133), we then have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla\rho_t\|^2 + \sigma\|\nabla_\Gamma\rho_t\|_\Gamma^2 + \int_\Omega f'(\rho)\rho_t^2 + \int_\Gamma g'(\rho)\rho_t^2\right) \\ + \|\nabla\mu_t\|^2 + \delta\|\nabla_\Gamma\mu_t\|_\Gamma^2 \\ \leq \frac{1}{2}\left|\int_\Omega f''(\rho)\rho_t^3 + \int_\Gamma g''(\rho)\rho_t^3\right| \\ \leq C(\|\rho_t\|^3_{L^3(\Omega)} + \|\rho_t\|^3_{L^3(\Gamma)}) \leq C\|\rho_t\|_V^3. \tag{3.135}$$

Since the term under the time derivative may be unbounded from below, due to the nonmonotonicity of f and g, we add to (3.135) the inequality

$$K\frac{\mathrm{d}}{\mathrm{d}t}\|\rho_t\|^2_{H^{-1}(\Omega)} \leq cK\|\rho_t\|_{H^{-1}(\Omega)}\|\rho_{tt}\|_{H^{-1}(\Omega)} \\ \leq c_\eta K^2\|\rho_t\|^2_{H^{-1}(\Omega)} + \eta\|\nabla\mu_t\|^2, \tag{3.136}$$

where the last passage follows by computing the H^{-1}-norm of ρ_{tt} by using (the time derivative of) Eq. (1.1). Then, taking $\eta > 0$ small enough and $K > 0$ large enough, the sum of the terms under time derivatives in (3.135) and (3.136) becomes nonnegative. Thus, recalling (3.31) and using the uniform Gronwall lemma, we obtain

$$\|\rho_t\|_{L^\infty(T_2,+\infty;V_\sigma)} \leq C, \tag{3.137}$$

e.g., for $T_2 = T_1 + 1$. Next, we consider (1.1) + (1.3) as a coupled time-dependent linear elliptic system, where, even in the worse case $\sigma = 0$, the forcing terms satisfy, thanks also to the last of (3.133),

$$\|\rho_t\|_{L^\infty(T_2,+\infty;V)} + \|w\rho_t\|_{L^\infty(T_2,+\infty;H^{1/2}(\Gamma))} \leq C. \tag{3.138}$$

Hence, by elliptic regularity (cf. [20, Lemma 2.2]) and standard bootstrap arguments, we have, at least,

$$\|\mu\|_{L^\infty(T_2,+\infty;H^2(\Omega))} + \delta^{1/2}\|\mu\|_{L^\infty(T_2,+\infty;H^3(\Omega))} \leq C. \tag{3.139}$$

We then turn to (1.2) + (1.4) and note that, even in the worse case $\sigma = \delta = 0$, the forcing terms satisfy

$$\|\mu - f(\rho)\|_{L^\infty(T_2,+\infty;V)} + \|w\mu - g(\rho)\|_{L^\infty(T_2,+\infty;V_\Gamma)} \leq C. \tag{3.140}$$

A further application of elliptic regularity results then yields (3.134). □

Next, let us consider a solution (ρ, μ) under the assumptions of Lemma 3.20. Due to this lemma, ρ eventually is precompact in $H^2(\Omega)$.

Then, we denote by $\mathbb{S}$ the family of the stationary solutions ρ_∞ satisfying the constraint (3.113). In the case of a singular function f, all elements of $\mathbb{S}$ are separated from the singularities, thanks to Lemma 3.17. Actually, one easily sees that the conclusions of Lemma 3.17 hold for all stationary solutions $(\rho_\infty, \mu_\infty)$ under the constraint (3.113) (and not only for those belonging to the ω-limit set of some weak solution). Moreover, deriving further estimates on the elements of $\mathbb{S}$ as in the proof of Lemma 3.20, it is not difficult to prove that $\mathbb{S}$ is bounded (at least) in $H^{5/2}(\Omega)$. Furthermore, the elements of $\mathbb{S}$ are the only stationary solutions which can belong to the ω-limit set of ρ. For simplicity, we restrict ourselves to the case

$m_0 = 0$ (otherwise, it would be sufficient to work on the translated trajectory $\rho - m_0$) and denote by $\mathbb{W}$ ($\mathbb{V}_\sigma$) the (closed) subspace of $H^2(\Omega)$ (V_σ, respectively) consisting of the functions having zero dm-average. Thus, ρ eventually belongs to $\mathbb{W}$ and also the elements of $\mathbb{S}$ have zero average.

We can then take an open neighborhood $\mathcal{U}$ of $\mathbb{S}$ in the $\mathbb{W}$-norm, chosen such that, in the case of a singular function f, any element of $\mathcal{U}$ is uniformly separated from the singularities. A simple contradiction argument then shows that ρ eventually takes values in $\mathcal{U}$.

We restrict the energy functional $\mathcal{E}$ to the set $\mathcal{U}$ and state a suitable version of the Simon–Łojasiewicz inequality, whose proof can be obtained by proceeding along the lines of [7, Prop. 6.6] (see also [9, Prop. 6.6]).

Lemma 3.21. *Let $\rho_\infty \in \mathcal{U}$ be a critical point of $\mathcal{E}$ and let F, G be analytic. Then, there exist constants $\theta \in (0, 1/2]$, $\Lambda > 0$, and $\iota > 0$ such that*

$$|\mathcal{E}(\rho) - \mathcal{E}(\rho_\infty)|^{1-\theta} \le \Lambda \|\mathcal{E}'(\rho)\|_{\mathbb{V}'_\sigma} \tag{3.141}$$

for all $\rho \in \mathbb{W}$ such that

$$\|\rho - \overline{\rho}\|_{\mathbb{W}} \le \iota, \tag{3.142}$$

where $\mathcal{E}'$ denotes the Fréchet derivative of the functional $\mathcal{E}$, restricted to $\mathcal{U}$, with respect to the norm of $\mathbb{V}_\sigma$.

Actually, we can be even more precise. Since $\mathbb{S}$ is bounded (at least) in $H^{5/2}(\Omega)$, it can be covered by a finite number of $H^2(\Omega)$-balls with arbitrarily small radius. This entails that $\mathcal{U}$ can be properly restricted in such a way that it still satisfies all the properties stated above and, in addition, the Simon–Łojasiewicz inequality (3.141) holds in the whole $\mathcal{U}$ with constants θ, Λ, ι which are independent of the point ρ_∞.

We are now ready to state the main result of this section.

Theorem 3.22. *Let the assumptions of Lemma 3.20 hold, let F and G be analytic, and let (ρ, μ) be a solution with $m_0 = 0$. Then, the ω-limit set of ρ consists of one single point.*

Proof. We first notice that, thanks to (3.40), the energy $\mathcal{E}(\rho)$ is a decreasing function of time and tends, as $t \nearrow +\infty$, to some value $\mathcal{E}_\infty$. Moreover, it is clear that $\mathcal{E}(\rho_\infty) = \mathcal{E}_\infty$ for all ρ_∞ in the ω-limit set.

Note that $\mathcal{U}$ is an open neighborhood of $\mathbb{S}$ (and, a fortiori, of the ω-limit set) in the $\mathbb{W}$-norm. Also, since ρ is precompact in $\mathbb{W}$, due to (3.134), a direct check shows that there exists some $T_3 > 0$ such that $\rho(t) \in \mathcal{U}$ for all $t \ge T_3$. Let us fix (an arbitrary) $t \ge T_3$. Then, we know that there exists ρ_∞ (a priori depending on the choice of t) such that (3.141) holds. We now compute, as in [7, Proof of Theorem 2.3] and for $t \ge T_3$, the derivative

$$\begin{aligned} -\partial_t(\mathcal{E}(\rho(t)) - \mathcal{E}_\infty)^\theta &= -\theta \partial_t(\mathcal{E}(\rho(t)) - \mathcal{E}_\infty)(\mathcal{E}(\rho(t)) - \mathcal{E}_\infty)^{\theta-1} \\ &\ge -\theta \frac{\partial_t \mathcal{E}(\rho(t))}{\Lambda \|\mathcal{E}'(\rho)\|_{\mathbb{V}'_\sigma}} \end{aligned} \tag{3.143}$$

and we can estimate the right-hand side as in [7].

Actually, on the one hand, we can take advantage of (3.40). On the other hand, $\mathcal{E}'(\rho)$ can be computed directly. Indeed, given any $k \in \mathbb{V}_\sigma$, a straightforward calculation leads to

$$\begin{aligned} \langle \mathcal{E}'(\rho), k \rangle_{\mathbb{V}_\sigma} &= \int_\Omega (\nabla \rho \cdot \nabla k + f(\rho)k) \\ &\quad + \int_\Gamma (\sigma \nabla_\Gamma \rho \cdot \nabla_\Gamma k + g(\rho)k) \\ &= \int_\Omega (-\Delta \rho + f(\rho))k \\ &\quad + \int_\Gamma (-\sigma \Delta_\Gamma \rho + g(\rho) + \partial_{\mathbf{n}} \rho)k. \end{aligned} \tag{3.144}$$

Notice that the integrations by parts are rigorous, since ρ eventually lies in $\mathbb{W}$.

Thus, using (1.2) and (1.4), we obtain

$$\langle \mathcal{E}'(\rho), k \rangle_{\mathbb{V}_\sigma} = (\mu, k)_m = (\mu - \mu_\Omega, k)_m. \tag{3.145}$$

Indeed, we can subtract μ_Ω (the "standard" average of μ over Ω), since $m(k) = 0$. Thus, using the Poincaré–Wirtinger inequality and passing to the supremum with respect to $k \in \mathbb{V}_\sigma$ with unit norm, we have

$$\|\mathcal{E}'(\rho)\|_{\mathbb{V}'_\sigma} \le c\|\mathcal{E}'(\rho)\|_{\mathcal{H}} \le c\|\mu - \mu_\Omega\|_{\mathcal{H}} \le c\|\nabla \mu\|. \tag{3.146}$$

Next, estimating the right-hand side of (3.143) with the help of (3.40) and (3.146), we infer the relation

$$\begin{aligned} -\partial_t(\mathcal{E}(\rho) - \mathcal{E}_\infty)^\theta &\ge \theta \frac{\|\nabla \mu\|^2 + \delta \|\nabla_\Gamma \mu\|_\Gamma^2}{\Lambda c \|\nabla \mu\|} \\ &\ge c(\theta, \Lambda)(\|\nabla \mu\| + \delta^{1/2} \|\nabla_\Gamma \mu\|_\Gamma), \end{aligned} \tag{3.147}$$

which holds for any $t \ge T_3$. Integrating over $(T_3, +\infty)$, we obtain

$$\nabla \mu \in L^1(T_3, +\infty; H), \qquad \delta^{1/2} \nabla_\Gamma \mu \in L^1(T_3, +\infty; H_\Gamma). \tag{3.148}$$

Now, from (1.1) and (1.3), we have, for any $\phi \in V_\delta$,

$$\langle \rho_t, \phi \rangle_{V_\delta} = (\rho_t, \phi)_m = \int_\Omega \nabla \mu \cdot \nabla \phi + \delta \int_\Gamma \nabla_\Gamma \mu \cdot \nabla_\Gamma \phi. \tag{3.149}$$

Thus, taking $\phi \in V_\delta$ with unit norm and passing to the supremum, it follows from (3.148) that

$$\rho_t \in L^1(T_3, +\infty; V'_\delta), \tag{3.150}$$

which shows that the *whole* trajectory ρ converges to a single element $\rho_\infty \in \mathbb{S}$, as desired. □

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