EXERCISE 7: sparse matrices (optional)

When the vast majority of a matrix A entries are zero, it is convenient to store only the nonzero values (and their position) in the memory. The MATLAB function **sparse** can convert a non-sparse (dense) matrix into a sparse one.

- Use the matrix A from Exercise 5. Using the MATLAB command **whos**, compare the memory usage when A is stored as dense and as sparse, for a large enough matrix size. Compare also the time spent to solve the system (use MATLAB LU factorisation, and MATLAB solver to invert the triangular systems).
- Consider a similar system $Bu = f$, where

$$B = \begin{bmatrix} 2 & -1 & -1 & \dots & -1 \\ -1 & 2 & 0 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ -1 & 0 & & & \\ \vdots & & & 2 & 0 \\ -1 & 0 & \dots & 0 & 2 \end{bmatrix}$$

  Note that A and B have the same number of nonzero entries, but different sparsity patterns. Choose $n > 5$ (the matrix B is singular for $n = 5$). Compare again the memory and solution time required when B is stored as sparse or as dense. Do you observe any difference with the previous case? If yes, why? You might want to compare the sparsity pattern (MATLAB command **spy**) of the L U factors.

EXERCISE 8

- Write MATLAB functions of the form `[x,res,its] = Jacobi(A,b,tol,maxit,x0)` that implement the Jacobi method. Do the same for the Gauss-Seidel method. Here `res` is the vector of residual norms for each iteration. Use this algorithm (stop when $\|r^{(k)}\| \leq \texttt{tol}\,\|r^{(0)}\|$ or $\texttt{its} \geq \texttt{maxit}$).
  Choose M, N with $A = M - N$ and compute $r^{(0)} = b - Ax^{(0)}$. If $r^{(0)} = 0$, return $x^{(0)}$ with zero iterations.
  For $k = 1, \ldots, \texttt{maxit}$, until the residual test holds:
  $x^{(k)} = x^{(k-1)} + M^{-1} r^{(k-1)}$ (do not compute the matrix M^{-1}, just solve the system)
  $r^{(k)} = b - Ax^{(k)}$
  Report success only if the residual test holds; otherwise report the iteration limit.
  Useful MATLAB commands: `M = diag(diag(A))`, `M = tril(A)`.
- Test the two methods on the problem $Ax = b$, where

$$A = \begin{bmatrix} 7 & 6 & 3 \\ 2 & 5 & -4 \\ -4 & -3 & 8 \end{bmatrix}, \qquad b = \begin{bmatrix} 16 \\ 3 \\ 1 \end{bmatrix},$$

  using $\texttt{tol} = 10^{-6}$, $\texttt{maxit} = 1000$, $\texttt{x0} = (0, \ldots, 0)^T$. Compare the history of convergence of the two methods, i.e. plot the iteration number vs. the norm of the residual. What is the value of $\max_i |\lambda_i(B)|$, where B is the iteration matrix?

EXERCISE 9

Consider the system

$$Ax = b$$

with $A =$ `gallery('poisson',n)` of size $n^2 \times n^2$ and $b = (1, \ldots, 1)^T \in \mathbb{R}^{n^2}$.

- For $n = 40$ solve the system using the Jacobi, Gauss-Seidel and CG (as implemented by the `pcg` function) methods. Use $x^{(0)} = 0$, the same tolerance and iteration limit for all methods; compare their residual histories and report convergence flags.

- For $n = 10, 20, \ldots, 100$ solve the system using CG. Check how the number of iterations varies, by plotting the number of iterations vs. n. Check also how the condition number $\lambda_{\max}(A)/\lambda_{\min}(A)$ varies (use the function `eigs`), and try to relate it with the number of iterations.