A KINETIC MODEL FOR MULTIDIMENSIONAL OPINION FORMATION

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Abstract. In this paper, we deal with a kinetic model to describe the evolution of the opinion in a closed group with respect to a choice between multiple options, e.g. political parties, which takes into account two main mechanisms of opinion formation, namely the interaction between individuals and the effect of the mass media. We provide an existence and uniqueness result for the model, and then we numerically test it in some relevant cases.

1. Introduction

The idea of modeling sociological behaviours by using tools of statistical mechanics arose about thirty years ago. The topics covered by such a research field, called *sociophysics* by Galam *et al.* in the early eighties [13], deal with several different problems, including social networks, population dynamics, voting, coalition formation and opinion dynamics.

Many different mathematical strategies have been explored. A very popular technique consists in using a description based on Ising models, see [11, 19, 12] for example.

The kinetic theory has been only in recent years applied to describe collective behaviour phenomena (see [15, 14, 16], for instance, as a possible introduction of Boltzmann-like equations in the context of sociophysics). During the last few years, the opinion formation with respect to a binary question (typically, a referendum), or in situations using a monodimensional opinion variable, has been modelled by kinetic-type equations in [20, 6, 8, 7, 10]. The main advantage of the kinetic formulation with respect to other strategies, such as the Ising models, is that one can also take into account intermediate opinions, therefore allowing to describe a partial agreement (or disagreement). It is worth noticing that the kinetic description has mainly been employed, up to now, in that case of binary questions.

Overtaking the situations concerning this kind of opinion formation is not completely straightforward from a modelling viewpoint. Indeed, a major problem consists in the fact that, with a plurality of possible options, in general, it is not possible to rank the options independently on the individual. For example, the schematization left/right in politics is not univocal, and some persons, although having a clear political orientation, can vote

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for another party which they do not share the same ideas with, but which defends some tangible interest that is very important to them.

The order between parties being a personal matter, it is hence not possible to use only one scalar independent opinion variable, at least when there are more than two political parties. It then becomes necessary to introduce an opinion vector, whose dimension coincides with the number of the possible choices. We note moreover that the opinion on an option can be independent on the opinion on the others ones: the use of a multidimensional opinion variable allows to take into account situations where the individual has a positive viewpoint for more than a possibility of choice.

In this paper, we give a specific attention to a particular problem of multidimensional opinion formation: we aim to tackle the modelling of a large population which must choose between two or more available political parties through a vote. However, our results are not limited to this context: our strategy can indeed be used in many other situations, for instance, the choice between some products in a no-monopolistic commercial market.

In a large-scale election, it is well accepted that two different processes play a fundamental role in the phenomenon of opinion formation inside a population: the binary exchange of ideas between individuals and the influence of the mass media (for example, TV networks, radios, newspapers, internet).

Whereas the interpersonal communication is always an essential ingredient in the time evolution of the public opinion, the interaction with media is typical of some kind of choices.

For example, in a small-scale election (such as a local election in a small community), the effect of the media, if any, is often less efficient than the interpersonal exchange of ideas between individuals: here, the direct knowledge of the candidates turns out to be decisive. Note that, in such a context, it would also be convenient to discuss the relevance of the kinetic description, since the basic assumption on the large size of the population cannot hold anymore.

This last phenomenon has been very well understood by political actors, who make a great effort in order to take advantage from it (e.g. by using direct advertisements, but also by controlling directly or indirectly the media themselves).

In this work, we have chosen to model the interpersonal communication by a unique mechanism, which is very similar to the process defined in [6]: the binary exchange between individuals induces concentration towards the majority opinion.

In general, this assumption does not hold. It is indeed clear that, in a real situation, the phenomenology is much more intricate. For example, many different behaviours of opinion formation depend on the fact that the way people think is not uniform. A realistic model should therefore include as many binary interaction rules between individuals as the mental paths of the members of the population. It is worth noticing that the variety of behaviours in the context of interpersonal communication is already enough to explain many interesting phenomena, such as the concentration towards some particular opinions or the cyclic (in time) behaviour of the distribution function: we refer to [10, 5] in the case of models depending on one scalar variable.

However, one of our primary objectives of the paper is to focus on the influence of the mass media on the population. Since the introduction of other factors which can modify the opinion evolution can hide the direct impact of the mass media, we shall therefore disregard them, and only consider a unique interaction rule coupled with the media effect. In this context, the binary collision rule which tends to the consensus seems the best choice, since it represents a very popular way of thinking and, in many populations, it is the most common behaviour [9, 1, 2, 3].

The next step in the modeling analysis should be to consider a more complete model, with a plurality of possible effects as the ones briefly mentioned above.

Concerning the characteristics of the media, we choose to consider the media opinion as an external input to the population. It means that someone, which is not influenced by the population itself, tries to modify the equilibrium which would be reached in a media-free situation. Of course, there are other possibilities. For example, people may prefer to be informed by media which share, more or less, their opinions. Hence, instead of releasing ideas without caring about the population, such kind of media may follow the local (in time) majority, in order to be more popular than the concurrence. It would mean that this kind of media would propose opinions which depend on the average opinion of the population.

Our choice for the media characteristics has the advantage of permitting the study of possible manipulation effects on the population. This kind of situations has been individuated and analyzed [17]: there exist, indeed, special interest groups which are able to manipulate the public opinion through the media, both in democratic societies and in autocratic ones.

In the earlier stages of development, for example, the state and the political parties tend to play an important role in financing the media (either directly through ownership, or indirectly through advertising). In more developped markets, even if the effects are more subtle, they still are present.

The model of the propaganda process is however complex [18]. It takes the form of a message flow through a network system, originated from an institution, and which can end with the possibility of response from the public. The message of the institution is carried by some propaganda agents (which can be charismatic people, but also bureaucrats or low-key disseminators of information). In order to successfully complete the process, the strength of the carrier is of course crucial.

Once established the aforementioned phenomena, the dynamics of their competition is clear. Whereas the binary exchange between individuals induces concentration towards a weighted majority opinion, the presence of the media disturbs this tendency to compromise through an attraction effect towards the media opinion.

We point out that our model describe the evolution of the opinion in a community with respect to a multidimensional choice, but it does not provide any forecast on the choice itself. By analogy with quantum mechanics, we might say that our model foresees the time evolution of the state of a system, whereas the choice process is the analogous of the measurement process, which gives, as a result, an eigenstate of the system itself.

Obviously, the starting point of the choice process is given by the distribution function, but it cannot only be determined by it. In order to justify this statement, let us consider, as an example, the electoral process. When voters only have two options, the strategy of vote seems quite clear. Eeach individual chooses the political party which is closer to (or at least not so far from) his opinion. Exceptions to this rule are quite uncommon, and are mainly due to people who think that the available parties are unsatisfactory.

On the contrary, when there are three or more options, the electoral system plays a crucial role, and the translation of the voter's sympathy into a vote heavily depends on the voting process. If the system is purely proportional, the strategy of vote can still be based on a maximal agreement rule. On the other hand, if the individuals are confronted to a majority system (or a mixed one), sometimes they would vote for a party which is not the best one in their own opinion, but that has real possibilities of winning the election, rather than a party which fits their opinion, but in no position of winning.

Here we choose to only consider a purely proportional voting system, which allows to establish a clear link between the opinion of an individual and its vote. Other systems of vote imply the coupling of our model of opinion formation with a strategy of vote based on game theory [15].

The paper is organized as follows. In the next section, we describe our model, written in a weak form with respect to the opinion vector. Then, in Section 3, we obtain an alternative formulation, the strong form, which is the starting point to study the main mathematical properties of the model. Existence and uniqueness are deduced in Section 4. Note that the number of available political parties is arbitrary, but it is obviously finite. We only performed numerical simulations for two and three parties, and the results are collected in Section 5. This choice is due to the fact that the distribution function can be visualized only when the available options are at most two. With more than two options, only quantities related to the distribution function (but not the distribution function itself) can be visualized. Moreover, the dynamical complexity of having more than two options is fully present in the 3D case.

2. Model for opinion formation

Let us consider an election process with $p \geq 1$ political parties, denoted as P_i, $1 \leq i \leq p$. For each party P_i, we introduce an *agreement variable* $x_i \in [-1, 1]$. In the following, Ω denotes the open interval $(-1, 1)$.

We label with $x_i = -1$ and $x_i = 1$ the two extreme behaviours: the complete disagreement with the party P_i is translated into the model by setting $x_i = -1$, and the opposite situation, *i.e.* the complete agreement, is translated by setting $x_i = 1$. Note that any intermediate value between the two extremes, $x_i = 0$ excluded, means a partial agreement or disagreement, with a degree of conviction proportional to $|x_i|$. The value $x_i = 0$ means a total indifference with respect to party P_i.

Since there are several parties, it can be useful to define the *opinion* (or *agreement*) *vector* $x = (x_1, \ldots, x_p) \in \bar{\Omega}^p$, which gives, for each individual of the population, its feelings about the political parties.

The unknown of our model is a density (or distribution function) $f = f(t, x) \geq 0$, defined on $\mathbb{R}_+ \times \bar{\Omega}^p$, whose time evolution is described by a kinetic-type equation. If the agreement vector is defined on a sub-domain $D \subseteq \bar{\Omega}^p$, the integral

$$\int_D f(t, x)\, \mathrm{d}x$$

represents the number of individuals with opinion included in D at time $t \geq 0$. Note that, in order to give a meaning to the previous considerations, f should satisfy $f(t, \cdot) \in L^1(\Omega^p)$ for all $t \in \mathbb{R}_+$.

As sketched in the introduction, we only take into account two processes of opinion evolution. The first one is given by the binary interaction between individuals, who exchange their points of view and adjust their opinions on the ground of each other's belief. The second one is the interaction with the media. Both phenomena are accurately presented below.

2.1. Exchange of opinions inside the population.

We model this process by borrowing the collisional mechanism of a typical interaction in the kinetic theory of gases: whereas, in rarefied gas dynamics, the particles exchange momentum and energy in such a way that the principles of classical mechanics are satisfied, here the "collision" between individuals allows the exchange of opinions.

Let $x, x^* \in \bar{\Omega}^p$ the opinion vectors of two individuals before an interaction. We suppose that the opinions after the interaction change according to the following rule:

$$(1) \qquad \begin{cases} x_i' = \dfrac{x_i + x_i^*}{2} + \eta(x_i)\, \dfrac{x_i - x_i^*}{2}, \\[2ex] (x_i^*)' = \dfrac{x_i + x_i^*}{2} + \eta(x_i^*)\, \dfrac{x_i^* - x_i}{2}, \end{cases} \qquad 1 \leq i \leq p.$$

Of course, other choices, based on sociological considerations, are possible. For $p = 1$, with (1), we recover the collision rule defined in [6].

The function $\eta : \bar{\Omega} \to \mathbb{R}$, which we henceforth name the *attraction function*, is smooth and it describes the degree of attraction of the average opinion with respect to the starting opinion of the agent. Note that η may depend on i, but we choose not to take into account this dependence, since we obtain the same kind of results. In the sequel, we need some more assumptions on the attraction coefficient η.

Definition 2.1. *Let $\eta : \bar{\Omega} \to \mathbb{R}$ be an even function of class $C^1(\bar{\Omega})$. The attraction function is* admissible *if*

- $0 \le \eta(s) < 1$ *for all* $s \in \bar{\Omega}$,
- $\eta'(s) < 0$ *for all* $s \in [-1, 0]$,
- *the Jacobian* $J(x_i, x_i^*)$ *of the collision mechanism* (1)*, taken component by component, i.e.*

$$J(x_i, x_i^*) = \frac{1}{2}[\eta(x_i) + \eta(x_i^*)] - \frac{1}{4}\eta'(x_i)\eta'(x_i^*)(x_i - x_i^*)^2$$
$$+ \frac{1}{4}[\eta'(x_i) - \eta'(x_i^*)](x_i - x_i^*) + \frac{1}{4}[\eta'(x_i)\eta(x_i^*) - \eta(x_i)\eta'(x_i^*)](x_i - x_i^*)$$

is uniformly lower bounded by a strictly positive constant, i.e. there exists $J_{\min} > 0$ such that $J(x_i, x_i^*) \ge J_{\min}$, for any i and any couple $(x_i, x_i^*) \in \bar{\Omega}^2$.

The first property prevents that the interaction destroys the bounds of the interval Ω. The second one translates the assumption that the effects of the interaction between individuals is stronger when the pre-collisional opinions are close to zero. The third one ensures that the inverse of the collision rule (1) is well defined.

Remark 2.2. *By using the properties listed in Definition* 2.1*, it is not difficult to also prove that, for any i and x_i, $x_i^* \in \bar{\Omega}$,*

$$x_i' - (x_i^*)' = \frac{1}{2}\left[\eta(x_i) + \eta(x_i^*)\right](x_i - x_i^*),$$

and, since $0 \le \eta < 1$,

$$|x_i' - (x_i^*)'| \le |x_i - x_i^*|.$$

It is then clear that the lateral bounds are not violated, i.e.

$$\max\{|x_i'|\,,\,|(x_i^*)'|\} \le \max\{|x_i|\,,\,|x_i^*|\}.$$

We note that the set of admissible attraction coefficients is not empty. A possible choice is $\eta(s) = \lambda(1 + s^2)$, with $0 < \lambda < 1/2$.

Once defined the collision rule (1), the interaction between individuals and the corresponding exchange of opinions is described by a collisional integral of Boltzmann type.

The collisional integral, which is denoted as Q, has the classical structure of the dissipative Boltzmann kernels. At a formal level, it can be viewed as composed of two parts: a *gain term* Q^+, which quantifies the exchanges of opinion between individuals which give, after the interaction with another individual, the opinion vector x, and a *loss term* Q^-, which quantifies the exchanges of opinion where an individual with pre-collisional opinion vector x experiences an interaction with another member of the population.

It is apparent that the existence of a pre-collisional pair which restitutes the post-collisional pair (x, x_*) through a collision of type (1) is not guaranteed, unless we suppose that the collisional rule is a diffeomorphism of $\bar{\Omega}^{2p}$ onto itself. Unfortunately, the collisional mechanism (1) does not verify this property. For instance, in general, there is no $(x_i, x_i^*) \in \bar{\Omega}^2$ which gives, after collision, the couple of extreme opinions $(-1, 1)$.

In order to overcome this difficulty, the natural framework for such a collision rule is given by the weak form. Two choices are possible. We may either build a model in a weak form with respect to x only, or work in a weak setting with respect to the whole set of independent variables. We choose the first option, which seems to be the correct framework for such kind of models.

A crucial term of the collision integral is given by the cross section. This quantity measures the probability of interaction between individuals and, moreover, the probability that the interaction causes a modification of the agent's opinion.

We suppose that the cross section $\beta : \Omega \times \Omega \to \mathbb{R}_+$ is a function of class $L^\infty(\Omega \times \Omega)$ which depends on a suitable pre-collisional opinion distance.

Let $\varphi = \varphi(x)$ be a suitably regular test function. We define the weak form of the collision kernel as

$$\langle Q(f,f), \varphi \rangle = \int_{\Omega^{2p}} \beta(x, x^*) f(t,x) f(t,x^*) \left[\varphi(x') - \varphi(x)\right] \, dx^* dx. \tag{2}$$

Note that the particular form of the collision rule (1) only enters through the test function $\varphi(x')$. It is also clear that the operator Q only acts on the agreement vector, and not on time.

The explicit form of the change of variables (1) also allows to give the following alternative formulations of the collision kernel:

$$\begin{aligned}
&\langle Q(f,f), \varphi \rangle \\
&= \int_{\Omega^{2p}} \beta(x, x^*) f(t,x) f(t,x^*) \left[\varphi((x^*)') - \varphi(x^*)\right] \, dx^* dx \\
&= \frac{1}{2} \int_{\Omega^{2p}} \beta(x, x^*) f(t,x) f(t,x^*) \left[\varphi(x') + \varphi((x^*)') - \varphi(x) - \varphi(x^*)\right] \, dx^* dx.
\end{aligned}$$

Remark 2.3. *At least formally, we have* $\langle Q(f,f), 1 \rangle = 0$.

2.2. **The influence of media.** The effects of the media on the population are here modelled by a fixed background. This assumption adds a linear kinetic term into our equations. We consider a set of $m \in \mathbb{N}^*$ media. For any media M_j, $1 \le j \le m$, we introduce two quantities: its *strength* α_j, which translates the influence of the media on the population and its opinion vector $X^j \in \Omega^p$, with respect to each political party.

Both quantities can be time-dependent. In what follows, we can suppose that the strength of the media is constant. This is the simplest assumption. It seems reasonable if the time scale is small enough, as it may happen during an electoral process. Of course, other choices, based on sociological considerations, are possible.

We do not suppose, however, that the opinion vector of the media is time-independent: even if, normally, the opinionists are quite stable in their convictions, some events can considerably modify the appealing of a party. Moreover, a particular strategy of manipulation of the public opinion, which is investigated in Section 5.1, is based on a time-evolution of the opinion vector of the media. Hence, in the following, we admit that $X^j : \mathbb{R}_+ \to \Omega^p$ for any j.

The effect of each media M_j on the individual is therefore described by an interaction rule which reminds the collision rule (1):

$$\tilde{x_i} = \Phi_i^j(x_i) = x_i + \xi_j(|X_i^j - x_i|)\ (X_i^j - x_i), \tag{3}$$

for all i and j.

The functions $\xi_j : [0,2] \to \mathbb{R}$ are the *influence functions* and satisfy the prescription collected in the following definition:

Definition 2.4. *Let $1 \le j \le m$ and $\xi_j : [0,2] \to \mathbb{R}_+$ be a function of class $C^1([0,2])$. The influence function is admissible if $0 \le \xi_j(s) < 1$, and if there exists $c_j \in (0,2)$ such that*

- *$\xi_j(s) = 0$ for all $s \in [c_j, 2]$,*
- *${\xi_j}'(s) < 0$ for all $s \in (0, c_j)$.*

Using this definition, we have the following proposition, whose proof is immediate.

Proposition 2.5. *The rule (3) is invertible. More precisely, the function $\Phi_i^j : x_i \mapsto \tilde{x_i}$ is a C^1-diffeomorphism on $\bar{\Omega}$ for any $j = 1, \dots, m$ and for any $i = 1, \dots, p$.*

The set of admissible influence functions is not empty. Indeed, a possible choice of ξ_j, with $c_j = 1$ and $0 < \lambda < 1/2$, is:

$$\xi_j(s) = \begin{cases} \lambda(1 + \cos(\pi s)) & \text{if } |s| \in [0,1], \\ 0 & \text{otherwise.} \end{cases}$$

The influence function acts in a different manner from the attraction function, since it depends on the distance between the opinion of the agent and the opinion of the media. When $\xi_j = 0$, the media has no effect in changing the opinion of the corresponding individual. This hypothesis translates the idea that the media may more easily influence people with a similar opinion.

Once defined the interaction rule (3), the influence of each media is described by a (possibly time-dependent) linear integral operator, L_j, $1 \le j \le m$, that has the classical structure of the linear Boltzmann kernels. Also in this case, the natural framework is the weak formulation.

Let $\varphi = \varphi(x)$ be a suitably regular test function. We define the weak form of the interaction kernel as

$$\langle L_j f, \varphi \rangle = \alpha_j \int_{\Omega^p} f(t,x)\,[\varphi(\tilde{x}) - \varphi(x)]\; dx. \tag{4}$$

Remark 2.6. *At least formally, we have* $\langle L_j f, 1\rangle = 0$.

2.3. **Combining the two phenomena.** We are then able to write down the whole model. Let $T > 0$. The evolution law of the unknown $f = f(t,x)$ results in an integro-differential equation:

$$\int_{\Omega^p} f_t(t,x)\varphi(x)\,dx = \sum_{j=1}^{m} \langle L_j f, \varphi\rangle + \langle Q(f,f), \varphi\rangle \tag{5}$$

posed in $(t,x) \in [0,T] \times \Omega^p$, for all $\varphi \in C(\Omega^p)$, with initial condition

$$f(0,x) = f^{\text{in}}(x) \text{ for all } x \in \Omega^p. \tag{6}$$

3. Alternative weak formulation

3.1. **Collision term.** The form of the collisional integral given by (2) is not completely satisfactory for the gain term because of the intricate dependence of the argument of the test function on the variables x, x_*. We therefore consider the weak form of the gain term

$$\langle Q^+(f,f), \varphi\rangle = \iint_{\Omega^{2p}} \beta(x,x^*) f(t,x) f(t,x^*)\varphi(x')\,dx^* dx. \tag{7}$$

In the same way as in [6], let us denote

$$D^i_\eta = \left\{ (x_i, x_i') \in \mathbb{R}\times\bar{\Omega} \,\middle|\, \frac{x_i'-1}{2} + \eta(x_i')\frac{x_i'+1}{2} \le x_i \le \frac{x_i'+1}{2} + \eta(x_i')\frac{x_i'-1}{2} \right\}$$

and

$$K_\eta(x,x') = \prod_{i=1}^{p} \frac{2}{1-\eta(x_i')}\chi_{D^i_\eta}(x_i,x_i'), \qquad \forall\, x, x' \in \bar{\Omega}^p,$$

where $\chi_{D^i_\eta}$ is the characteristic function of the set D^i_η. Since η is an admissible attraction function, it is clear that $D^i_\eta \subseteq \bar{\Omega}^2$. Note that, for a fixed η, K_η is obviously of class $L^\infty(\Omega^{2p})$.

We then perform the change of variables $x^* \mapsto x'$ in (7), for a fixed x. It is easy to see that

$$dx_i^* = \frac{2}{1-\eta(x_i)}dx_i' \qquad \text{and} \qquad x_i^* = \frac{2x_i' - x_i - \eta(x_i)x_i}{1-\eta(x_i)}.$$

Then, after permuting x and x', we obtain the following weak form of the gain term:

$$\langle Q^+(f,f), \varphi\rangle = \iint_{\Omega^{2p}} \beta(x',y) K_\eta(x,x') f(t,y) f(t,x')\varphi(x)\,dx' dx,$$

where

$$y_i = \frac{2x_i - x_i' - \eta(x_i')x_i'}{1-\eta(x_i')}, \qquad 1 \le i \le p.$$

A new weak form of the collision operator immediately comes:

$$\langle Q(f,f),\varphi\rangle = \langle Q^+(f,f),\varphi\rangle - \iint_{\Omega^{2p}} \beta(x,x')f(t,x)f(t,x')\varphi(x)\,dx'dx,$$

which becomes the definition of the collisional kernel in our model.

3.2. **Linear term.** We can obtain a similar result for the operator which models the interactions with a media. For a given j, we use again the notation Φ_i^j introduced in Proposition 2.5. Let us then denote Ψ_i^j the inverse function of Φ_i^j, and successively set, for any $z \in \bar{\Omega}^p$,

$$\Psi^j(z) = (\Psi_i^j(z_i))_{1\le i\le p}, \qquad R^j(z) = \left(\prod_{i=1}^p (\Phi_i^j)'[\Psi_i^j(z_i)]\right)^{-1}.$$

Since Φ^j is a C^1-diffeomorphism, there exists $R>0$ such that the nonnegative function R^j is upper bounded by R. We can now perform the change of variables $x \mapsto \tilde{x}$ in (4), permute x and $\tilde{x}$, and obtain a new weak form for the media action:

$$\langle L_j f,\varphi\rangle = \alpha_j \int_{\Omega^p} f(t,\Psi^j(x))\; R^j(x)\; \varphi(x)\; dx - \alpha_j \int_{\Omega^p} f(t,x)\; \varphi(x)\; dx.$$

4. Existence and uniqueness theorem

In this section, we aim to study some mathematical properties of our problem (5)–(6). We first obtain some a priori estimates, and then deduce a theorem which asserts the existence and uniqueness of a solution to (5)–(6).

Our model guarantees the conservation of the total number of individuals of the population. By borrowing the kinetic theory language, the following result is also named the total mass conservation.

Proposition 4.1. *Let $f = f(t,x)$ be a nonnegative solution of* (5)–(6)*, with a nonnegative initial condition $f^{\mathrm{in}} \in L^1(\Omega^p)$. Then we have*

$$\|f(t,\cdot)\|_{L^1(\Omega^p)} = \|f^{\mathrm{in}}\|_{L^1(\Omega^p)}, \qquad \text{for a.e. } t \ge 0.$$

Proof. We simply consider Equation (5) with test function $\varphi \equiv 1$. □

Of course, the mass conservation is not realistic if we consider long-time forecasts. Indeed, in such situations, we should also consider processes of birth, death and shift of age of the voters, which would lead to the variation of the total number of individuals. But usually, as in the case of elections or referendums, the interest of such models is to deduce short-term forecasts by using, as an initial datum, the result of some opinion poll.

Since $|x| \le 1$, from the mass conservation, we immediately deduce, in the following corollary, that all the moments of f are bounded.

Corollary 4.2. *Let $f = f(t,x)$ be a nonnegative solution of (5)–(6), with nonnegative initial condition $f^{\text{in}} \in L^1(\Omega^p)$. Then, for any $n \geq 1$,*

$$\int_{\Omega^p} x^n f(t,x)\,dx \leq \|f^{\text{in}}\|_{L^1(\Omega^p)}, \qquad \text{for a.e. } t \geq 0.$$

We are now ready to prove the existence of weak solutions to our problem. The results are collected in the following theorem:

Theorem 4.3. *Let f^{in} a nonnegative function of class $L^1(\Omega^p)$. Then, for all $T > 0$, Equation* (5)–(6) *admits a unique nonnegative solution $f \in C^0([0,T]; L^1(\Omega^p))$.*

Proof. Let $T > 0$. We consider the operator $\Theta : f \mapsto \Theta f$ defined on $C^0([0,T]; L^1(\Omega^p))$, for $t \in [0,T]$ and $x \in \Omega^p$, by

$$\begin{aligned} \Theta f(t,x) &= f(0,x) - \int_0^t \int_{\Omega^p} \beta(x,x')\ f(s,x)\ f(s,x')\ dx'\ ds \\ &\quad + \int_0^t \int_{\Omega^p} \beta(x',y)\ K_\eta(x,x')\ f(s,y)\ f(s,x')\ dx'\ ds \\ &\quad + \sum_{j=1}^m \alpha_j \int_0^t \big(f(s,\Psi^j(x))\ R^j(x) - f(s,x)\big)\ ds, \end{aligned}$$

where

$$y_i = \frac{2x_i - x_i' - \eta(x_i')x_i'}{1-\eta(x_i')}, \qquad 1 \leq i \leq p.$$

It is a direct consequence from Proposition 4.1 that $\Theta : C^0([0,T]; L^1(\Omega^p)) \to C^0([0,T]; L^1(\Omega^p))$.

The existence and uniqueness of a solution to (5)–(6) follow, if we can prove that Θ is a contraction in the functional space $C^0([0,T]; L^1(\Omega^p))$.

Indeed, (5) can be rewritten under a strong integral form as $\Theta f = f$. Let us hence consider u, $v \in C^0([0,T]; L^1(\Omega^p))$ sharing the same initial condition f^{in}. We have

$$\begin{aligned} &\|\Theta u - \Theta v\|_{L^\infty(0,T;L^1(\Omega^p))} \\ &\leq T \left[2\|\beta K_\eta\|_{L^\infty(\Omega^{2p})} \|f^{\text{in}}\|_{L^1(\Omega)} + (1+R^p) \sum_{j=1}^m \alpha_j \right] \|u-v\|_{L^\infty(0,T;L^1(\Omega^p))}. \end{aligned}$$

The quantity inside the square brackets in the previous inequality is a constant $A > 0$ which only depends on the data. Hence, if we choose $T_0 = (2A)^{-1} > 0$, Θ is a contraction on $[0,T_0]$. Then there exists a unique $f \in L^\infty(0,T_0; L^1(\Omega^p))$ such that $\Theta f = f$. Thanks to the expression of Θf, it is then clear that, in fact, $f \in C^0([0,T_0]; L^1(\Omega^p))$.

Moreover, since the mass is conserved by Proposition 4.1, we can apply a bootstrap method by using as initial datum $f(T_0,x)$, and extend, if necessary, the time interval up to $[0, T]$. By induction, the existence and uniqueness of a solution to (5)–(6) are proved. □

5. Numerical simulations

This section is devoted to investigate the numerical behaviour of the model. We limit ourselves to the 2D and 3D situations, mostly for computational cost and readability reasons. We apply the model to an electoral competition but, as explained in the introduction, other situations with an analogous dynamics (e.g. the choice between some products advertised by media) can be described by the same tool.

The computations are performed using a numerical code written in C. We consider a regular subdivision $(x^0, \dots, x^N)$ of Ω, with $N \geq 1$. The function f is computed at the center of each cubic cell $\prod_{i=1}^{p}(x^{k_i}, x^{k_i+1})$, $0 \leq k_i \leq N-1$. In our computations, we choose $N = 100$.

The scheme itself conserves the total agents number, i.e. $\|f(t)\|_{L^1_x}$. In order to numerically simulate collisions, we used a slightly modified Bird method [4]. Note that our scheme does not allow the scalar opinions to go out from $[-1, 1]$. As a matter of fact, opinions x such that $|x| > 1$ are not possible because the collision mechanism prevents them, and the media opinions of the media also live in $[-1, 1]$.

In the whole section, the form of the attraction function is $\eta(s) = 0.25(1+s^2)$. The influence function (independent on the media) is $\xi(s) = 0.9(1-s^2)$ on $[0, 1]$ and $\xi(s) = 0$ on $[1, 2]$. Note that, of course, this function ξ is not C^1, but that does not matter for the numerics. We also choose a constant cross section $\beta = 1$. The values of (α_j) are given as proportional to β.

5.1. **Two-party system.** We first choose $p = 2$, i.e. the phenomenon of opinion formation only concerns two political parties. Using a scalar opinion variable would then be an option, but, in this case, the meaning of the scalar variable would be the signed difference of the two components of the two-dimensional opinion vector. In fact, the 2D model contains more information about the population opinion than the 1D one. Anyway, we are also interested in the following integrals, which represent the population percentage respectively in favour of parties P_1 and P_2:

$$I_1 = \iint_{E_1} f(x)\, dx, \qquad I_2 = \iint_{E_2} f(x)\, dx = 1 - I_1,$$

where

$$E_1 = \{(x_1, x_2) \in \Omega^2 \mid x_1 > x_2\}, \qquad E_2 = \{(x_1, x_2) \in \Omega^2 \mid x_1 < x_2\}.$$

If not precised, the population is uniformly distributed: $f^{\text{in}}(x_1, x_2) = 0.25$.

5.1.1. *Media-free population.* We first treat the case of a population without media, whereas the dynamics is given only by the binary interactions between individuals, which should lead to a compromise effect. In Figure 1,

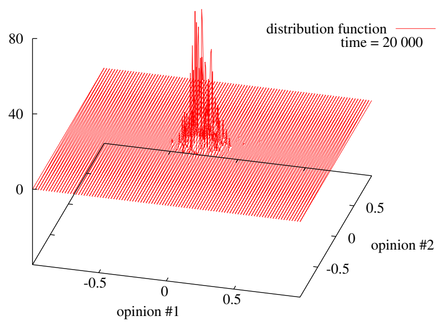

FIGURE 1. Distribution function at time $t = 20\,000$

we can observe that, at time $t = 20\,000$, the distribution function began to strongly get close to a Dirac mass centred in $(0, 0)$, as expected.

5.1.2. *Population away from the media influence.* We now consider the effect of one media whose opinion is too far from the opinion of the population. The initial datum $f^{\text{in}}(x_1, x_2) = 4$ when $(x_1 < -0.5, x_2 < -0.5)$ and zero otherwise. The opinion of the media is centred in $(0.9, 0.9)$, and its strength is $\alpha_1 = 0.1\beta$.

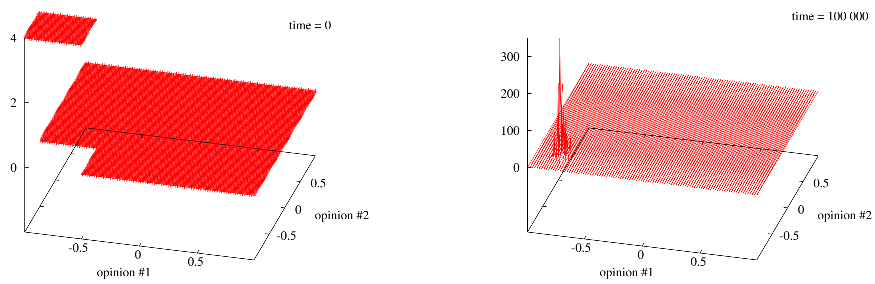

FIGURE 2. Distribution function at time $t = 0$ and $t = 100\,000$

We observe no effect of the media opinion on the population. The distribution function converges towards the Dirac mass centred in $(-0.75, -0.75)$. Figure 2 shows the plots of f at initial time and $t = 100\,000$.

5.1.3. *Influence of one unique media.* We here consider one unique media which can act on the population. When the media opinion is in agreement with the opinion of a part, even very small, of the population, its effect is far from being negligible.

The behaviour of the model is indeed obtained through the combined effect of compromise effect and influence of the media, that acts as a linear Boltzmann kernel.

Heuristically, the dynamics is the following. When the members of the group with precollisional opinions close to the media opinion interact with the remainder of the population, the opinions of the latters are drawn up towards the media opinion. In this case, a successive interaction with the media can have a significant effect, and the convergence to the media opinion becomes possible.

The model exhibits a threshold effect: if the fraction of the population whose interaction with the media is significant is above a critical value, then we can numerically recover that, asymptotically, the distribution function goes to a Dirac mass centred at the same point of the opinion of the media. Otherwise, the concentration effect of the media is not enough to draw the whole population to the media opinion.

We recover both behaviours in the next two numerical simulations. The media strength is set to $\alpha_1 = 0.1\beta$, and its opinion vector is $(0.9, 0.9)$.

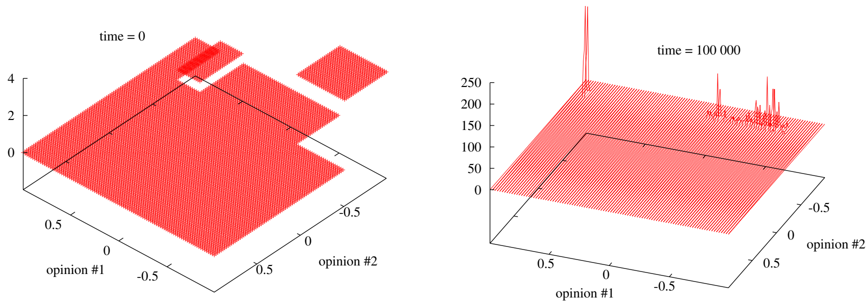

FIGURE 3. Distribution function for situation 1 at (a) $t = 0$ and (b) $t = 100\,000$

Situation 1. We consider an initial datum such that $f^{\text{in}}(x_1, x_2) = 56/15$ when $(x_1 < -0.5, x_2 < -0.5)$, $f^{\text{in}}(x_1, x_2) = 8/15$ when $(0.75 > x_1 > 0.5, x_2 < -0.5)$ and zero otherwise. In Figure 3, we observe that the concentration effect is not global: the distribution function has a Dirac-like behaviour in $(0.9, -0.75)$, but it cannot vanish around the region $(x_1 < -0.1, x_2 \simeq -0.75)$, because of the threshold on the media effect.

Situation 2. In this case, the initial datum satisfies: $f^{\text{in}}(x_1, x_2) = 32/9$ when $(x_1 < -0.5, x_2 < -0.5)$, $f^{\text{in}}(x_1, x_2) = 16/9$ when $(0.75 > x_1 > 0.5, 0.75 > x_2 > 0.5)$ and zero otherwise. This time, an almost full concentration effect around the media opinion is shown in Figure 4.

5.1.4. *Competition between two fixed unbalanced media.* From now on, we only use a population with a uniformly distributed opinion. We study the effect of two media with different strength. More precisely, we have $\alpha_1 = 0.1\beta$, $X^1 = (0.6, -0.4)$, $\alpha_2 = 0.3\beta$ and $X^2 = (-0.3, 0.7)$. In Figure 5, we

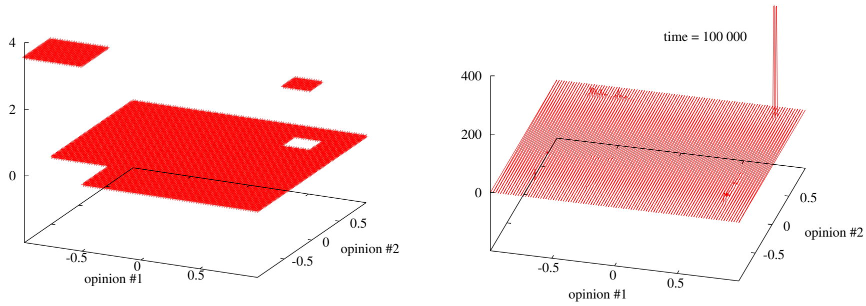

FIGURE 4. Distribution function for situation 2 at (a) $t = 0$ and (b) $t = 100\,000$

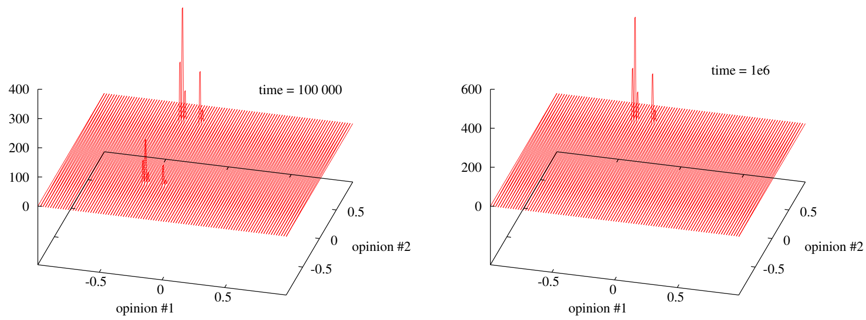

FIGURE 5. Distribution function with two media ($t = 100\,000$ and $1\,000\,000$)

observe the forming of two Dirac masses-like in X^2 and in $(-0.15, 0.7)$, and the vanishing of two other Dirac masses-like. The highest one is centred on the stronger media opinion. The remaining other one does not vanish when time grows, and its mass is one third of the one centred in X^2.

5.1.5. *Opinion manipulation by a media.* We choose the media with the same strength $\alpha_1 = \alpha_2 = 0.1\beta$, and the opinion vector of media M_1 is fixed $X^1 = (0.4, -0.4)$. Up to time $10\,000$, we compare the behaviour of the distribution function and of I_1 in the two following cases. We first choose $X^2 = (-0.4, 0.4)$, constant with respect to time. Then we choose X^2 with the same opinion, except when $3000 < t < 7000$, where $X^2 = (-0.39, 0.39)$.

One can check that, with these two choices, the distribution function centers on $(0, 0)$ when time grows, but does not become a Dirac mass. However, the impact of a variable media opinion is very strong with respect to time. Figure 6 shows that the variation of I_1 is violent, and that the result of the poll, which was previously balanced, is suddenly artificially moved in favour of party P_1.

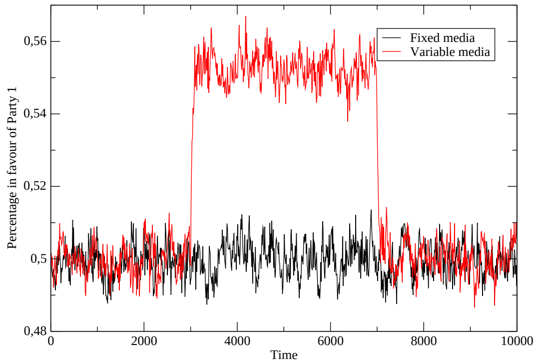

FIGURE 6. Plot of I_1 with respect to t using a fixed/variable media

From a sociological point of view, the model forecasts a growth of positive opinions concerning party P_1, induced by a very small change of X^2 towards X^1, without reinforcing the opinion of media M_1.

5.1.6. *A unique media or two media with half strength to represent the media opinion.* We compare the behaviour of the distribution function in two similar cases.

Situation 3. Three media act on the population, with the following characteristics:

$$X^1 = X^2 = (0.4, -0.4), \ X^3 = (-0.4, 0.4), \qquad \alpha_1 = \alpha_2 = 0.1\beta, \ \alpha_3 = 0.2\beta.$$

Situation 4. Two media act on the population, with the following characteristics:

$$X^1 = (0.4, -0.4), \ X^2 = (-0.4, 0.4), \qquad \alpha_1 = \alpha_2 = 0.2\beta.$$

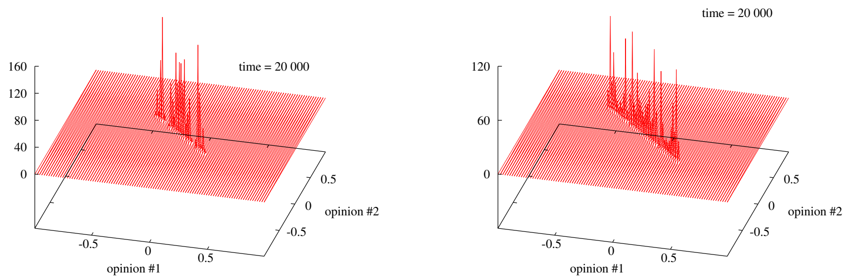

FIGURE 7. Distribution functions for situations 3 and 4 at $t = 20\,000$

One can check that the amount of people in favour of party P_1 oscillates around 0.5 in both situations, and one cannot draw any conclusion from this only information.

We note that media X^1 and X^2 of situation 3 have the same opinion as media X^1 of situation 4, the sum of the strength of the former being equal to the strength of the latter.

At a theoretical level, since the interaction with media is described by a linear term of the model, the results of situations 3 and 4 should be identical.

We can recover this feature in Figure 7: the supports of the two distribution functions are numerically identical and, moreover, their global shapes are similar. The differences in the shapes are originated by a numerical effect due to the treatment of the collisional part with a Bird method, based on a random routine.

5.1.7. *One strong media against two weaker ones.* We here investigate the situation of three media where their respective opinion vectors are

$$X^1 = (0.6, -0.6),\ X^2 = (-0.6, 0.6),\ X^3 = (-0.2, 0.2),$$

and their strength

$$\alpha_1 = 0.2\beta,\ \alpha_2 = \alpha_3 = 0.1\beta.$$

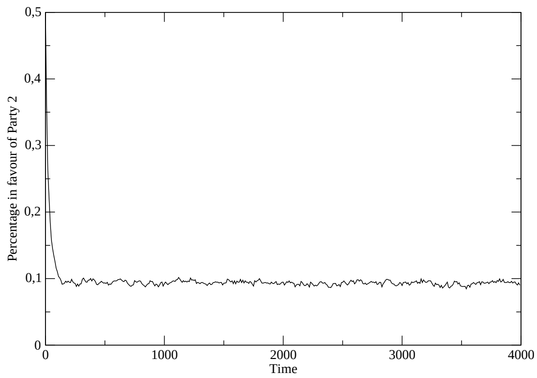

FIGURE 8. Plot of I_2 with respect to t when the media rather favour party P_1

As expected, we can see in Figure 8 that party P_2 is immediately weakened by the media influence.

5.2. **Three-party system.** We now choose $p = 3$, i.e. the phenomenon of opinion formation involves three political parties. Using a scalar opinion variable is not anymore an option.

Let

$$\begin{aligned} E_1 &= \{x \in \Omega^3 \mid x_1 > \max(x_2, x_3)\}, \\ E_2 &= \{x \in \Omega^3 \mid x_2 > \max(x_1, x_3)\}, \\ E_3 &= \{x \in \Omega^3 \mid x_3 > \max(x_1, x_2)\}. \end{aligned}$$

In the same way as in 5.1, we are interested in the three following integrals

$$I_1 = \iiint_{E_1} f(x)\, dx, \qquad I_2 = \iiint_{E_2} f(x)\, dx, \qquad I_3 = \iiint_{E_3} f(x)\, dx,$$

which can be interpreted in terms of percentage of the population more likely to vote, in a proportional system, for parties P_1, P_2 and P_3 respectively. In all the tests, the population is uniformly distributed: $f^{\rm in}(x) = 0.125$.

5.2.1. *A medialess party.* We investigate the situation where two of the three parties are supported by two media with the same strength, and the last one has no mediatic support. More precisely, we have

$$X^1 = (0.4, -0.4, -0.4), \; X^2 = (-0.4, 0.4, -0.4), \quad \alpha_1 = \alpha_2 = 0.1\beta.$$

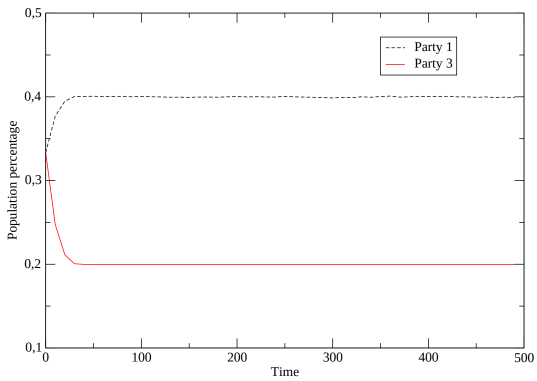

FIGURE 9. Plots of I_1 and I_3 with respect to t when party P_3 has no media support

Although it does not benefit from any mediatic help, supporters of party P_3 do not disappear, as one can check in Figure 9. Of course, he is weakened regarding the other parties, but still 20 % of the population may eventually vote for it.

5.2.2. *An extremist media.* We here consider a situation where party P_1 has a supporting media with a more asserted opinion, i.e. $X^1 = (0.9, -0.2, -0.2)$. The two other media support parties P_2 and P_3 with a more centred opinion, i.e. $X^2 = (-0.2, 0.3, -0.2)$ and $X^3 = (-0.2, -0.2, 0.3)$. The strength of each media is set to 0.1β.

The quantity I_2 is not plotted in Figure 10, the two curves of I_2 and I_3 are almost superimposed: there is a total symmetry between the second and third variables. In this situation, we can see that an extremist media does not really help the party which it supports: the moderate ones are far more efficient. Party P_1 has indeed the same result as if there were no media supporting it.

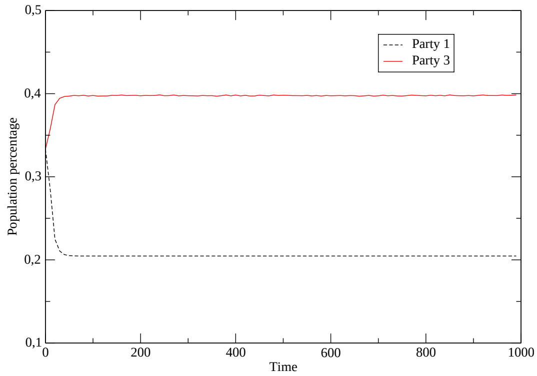

FIGURE 10. Plots of I_1 and I_3 with respect to t when party P_1 has an extremist support

5.2.3. *A strong media with a variable opinion.* Eventually, we study the behaviour of our model in the case when there is one media stronger than the other ones and whose opinion varies with respect to t. More precisely, we set

$$X^1(t) = (0.3, -0.2, -0.2) + 0.2\cos(2\pi t/100)(-1, 1, 1),$$
$$X^2 = (-0.2, 0.3, -0.2), \ \ X^3 = (-0.2, -0.2, 0.3),$$

and impose

$$\alpha_1 = 0.2\beta, \qquad \alpha_2 = \alpha_3 = 0.1\beta.$$

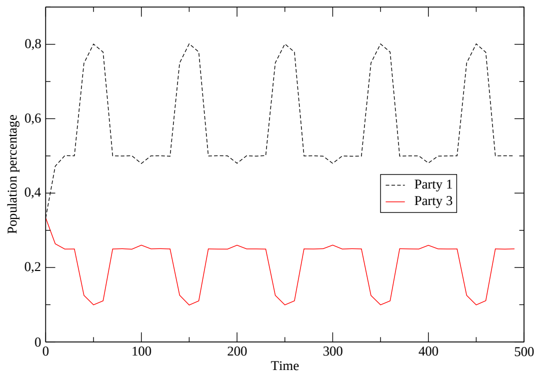

FIGURE 11. Plots of I_1 and I_3 with respect to t when party P_1 has a strong mediatic support

Once again, the quantity I_2 is not plotted in Figure 11, for symmetry reasons. The strength of media M_1, which is linked to party P_1, significantly increases the influence of this party. Moreover, the variations of X^1 induce some non vanishing oscillations on I_1 and I_3, as seen in Figure 11. When X_1^1 is close to its maximal value, around $t \equiv 50[100]$, the proportion of the population which favours party P_1 is around 80 %, whereas it should be around 50 %. On the contrary, when X_1^1 is close to its minimal value, party P_1 loses its absolute majority.

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