# CONCILIATORY AND CONTRADICTORY DYNAMICS IN OPINION FORMATION

LAURENT BOUDIN, AURORE MERCIER, AND FRANCESCO SALVARANI

Abstract. In this article, we use a kinetic description to study the effect of different psychologies on the evolution of the opinion with respect to a binary choice, in a closed group. We show that the interaction between individuals with different reactions regarding the exchange of opinion induces some phenomena, such as the concentration of opinions or the cyclic-in-time behaviour of the distribution function. We provide an existence and uniqueness result for the model and numerically test it in some relevant cases.

## 1. Introduction

The kinetic approach in sociophysics is a promising line of research to explain collective behaviours in a simple but mathematically solid way. Indeed, the methods of nonequilibrium statistical mechanics, classically used in the kinetic theory of gases, can also be fruitfully applied to study the collective behaviour of a large enough number of individuals, where none of them has a preponderant role with respect to the others.

A kinetic model consists of a set of partial integro-differential equations governing the time evolution of probability density functions, which fully describe the system. The independent variables of the unknown functions are the time and any other physical quantities which are relevant to the problem. For instance, when dealing with rarefied gases, the typical independent variables are the position and velocity of the gas molecules. In opinion dynamics, a common independent variable is the opinion – or the agreement – with respect to a binary question.

The introduction of kinetic models in the sociophysical literature started two decades ago [17, 16, 18]. This methodology has recently experienced a renewal of attention [19, 1, 6, 7, 9, 11, 5] for many reasons, see also the review article [8].

In this article, we study the time evolution of the opinion, with respect to questions of binary type (e.g. a referendum), in a closed community. This problem is a classical issue of sociophysics, and many authors have considered it, see [10, 15, 19, 9, 6, 7, 5], for example.

In our model, based on a kinetic approach, the unknowns are probability density functions which depend on two independent variables: time and

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opinion of the agents regarding the aforementioned binary question. The opinion of each individual can only be modified through the binary exchange of ideas with another member of the community.

Nevertheless, even if we only take into account this elementary phenomenon of opinion exchange, the collective behaviour of the population with respect to the binary question is far from being simple. Indeed, it is well accepted in the literature that many different behaviours concerning the dynamics of opinion formation depend on the fact that the way people think is not uniform. A realistic model should therefore include as many binary interaction rules between individuals as mental paths inside the population.

For simplicity reasons, we only consider here two ways of thinking, and even this simplified situation leads to interesting phenomena.

The first psychological attitude is typical of individuals which tend to compromise after an opinion exchange. This behaviour is widely recognized, and it can be considered as the most common in the literature [2, 3, 15, 10]. Besides, it is also the main common feature of the kinetic approaches, as it is emphasized in [8].

The other mental path is completely different, and leads to psychological dynamics of contradictory type. This behaviour has been proposed by Galam in [13, 14], and is based on the fact that some people are deliberately opposed to the choice of the interlocutors, whatever that choice may be.

These two psychological behaviours are translated in our paper by different kinds of interaction rules of kinetic type, which allow to obtain, in a deterministic way, the post-interaction opinions from the pre-interaction ones.

It is worth noting that the collision rules proposed here are only examples of possible behaviours. Actually, the psychology of an individual cannot be considered as a mechanical system and it may vary interaction by interaction. It is moreover clear that, in a real situation, the phenomenology is much more intricate (the effect of mass media can be decisive, for example [5]). Nevertheless, as we shall see, the presence of only two fixed behaviours in the context of interpersonal communication is already enough to explain many interesting phenomena, such as the concentration towards some particular opinions or the cyclic (in time) behaviour of the distribution function, two phenomena that have been sociologically observed [14].

In our model, we suppose that the population is closed. It means that the total number of individuals is constant. This assumption does not imply a great limitation, since the characteristic time of opinion evolution is very small with respect to the typical characteristic time in population dynamics. Moreover, we suppose that the probability of a binary interaction is constant. In a structured society, this hypothesis is not true. Indeed, people are normally involved in a social network, and hence some interactions are much more probable than other ones. However, we cannot explain the influence of a network as a function of opinion and time. Since we restrain ourselves to consider only those two independent variables, it seems logical to treat, as a first approximation, the probability of a binary interaction as a constant. We finally note that our description is well adapted only when the size of the population is large enough: this is the key-point of statistical mechanics.

All the previous assumptions have a double effect. On the one hand, they reduce the applicability of the model, on the other hand, by simplifying the phenomena taken into account, they allow to build a model which is tractable from a mathematical point of view.

The organization of the article is the following. We first describe our model, and study different interaction rules for both groups of agents. Then we prove an existence result for the considered problem. Eventually, we present some relevant numerical tests and provide an analysis of the quantitative results from a sociological point of view.

## 2. The kinetic model

In this paper, we study the opinions regarding a binary question (e.g. a referendum) in a non homogeneous population composed by individuals with different psychologies. We aim to estimate how different possible reactions with respect to the opinion exchange can influence its global behaviour.

In the following, $\Omega$ denotes the open interval $(-1, 1)$. We describe the opinion by means of a continuous variable $x \in \bar{\Omega}$, where $x = -1$ and $x = 1$ identify the two extreme positions. Any intermediate value between those values means that the corresponding individual partially agrees with the opinion labelled with the same sign, with a degree of conviction which is proportional to $|x|$. If $x = 0$, the corresponding individual has no preference with respect to the question.

We describe the population by using the kinetic approach. The main tool of the model is the concept of distribution function, a quantity which depends on the time $t$, on the opinion variable $x$, and on the features of the population. Its time evolution, governed by a partial differential equation, then allows to forecast the behaviour of the system. Since the population is non-homogeneous, a possible strategy of description consists in stratifying the individuals with respect to their psychological reactions during the opinion exchange process.

We assume that the members of the population belong to two different groups, and introduce two associated distribution functions which separately describe each group: the population can be split into *conciliatory* people and *contradictory* ones (the precise definitions are given later). This assumption is the simplest one that allows to investigate the effects of different psychologies on the global behaviour of the population. A generalization of the kinetic approach to more complex situations is obviously possible. In that case, the distribution function is of vectorial type, with as many entries as the possible psychological behaviours inside the population. Note that we do not pay attention to the spatial structure of the closed community, which is assumed interlinked. The two groups are respectively described by the nonnegative distribution functions $f := f(t,x)$ and $g := g(t,x)$. Both are defined on $\mathbb{R}_+ \times \bar{\Omega}$.

If $D$ is a subdomain of $\bar{\Omega}$, the integrals

$$\Big(\int_D f(t,x)\,\mathrm{d}x\Big)/\Big(\int_\Omega f(t,x)\,\mathrm{d}x\Big) \quad \text{and} \quad \Big(\int_D g(t,x)\,\mathrm{d}x\Big)/\Big(\int_\Omega g(t,x)\,\mathrm{d}x\Big)$$

represent the fraction of conciliatory and contradictory individuals with opinion included in $D$ at time $t$. Note that, in order to give a sense to the previous considerations, we need that $f(t,\cdot) \in L^1(\Omega)$ and $g(t,\cdot) \in L^1(\Omega)$ for all $t \in \mathbb{R}^+$. Those properties may not be satisfied but only asymptotically in time, as we shall see: when $t$ tends to $+\infty$, $f(t,\cdot)$ and $g(t,\cdot)$ may not remain in $L^1(\Omega)$ and become measures. However, in Theorem 1, we shall prove that the solution remains in $L^1(\Omega)$ for all finite time. Hence, the previous integrals have a meaning for all $t \in (0,T)$, for all $T > 0$.

As sketched in the introduction, we only take into account one process of opinion formation given by the interaction between agents, who exchange their point of view and influence themselves. Moreover, we suppose that the interactions between individuals are only of binary type. Multiple interactions can be seen as the result of a chain of binary exchanges.

We model this binary process by borrowing the collisional mechanism of a typical interaction in the kinetic theory of gases: whereas in rarefied gas dynamics, the particles exchange momentum and energy in such a way that the principles of classical mechanics are satisfied, here the interactions between individuals allow the exchange of opinions. Since there are two categories of people within the population, we define three types of interactions. We assume that the collision mechanisms do not destroy the bounds of the interval $\bar{\Omega}$. We shall detail the collision rules in Section 4.

Each post-interaction opinion can be written in terms of pre-interaction opinions and depends on the psychologies of the individuals. The interactions between individuals are described by a collisional integral of Boltzmann type, which has the classical structure of a dissipative Boltzmann kernel. Each collisional integral can be viewed as composed of two parts: a *gain term*, which quantifies the exchanges of opinion between individuals which give, after the interaction with another individual, the opinion $x$, and a *loss term*, which quantifies the exchanges of opinion where an individual with pre-interaction opinion $x$ experiences an interaction with another member of the population.

It is apparent, in general, that the existence of a pre-collisional pair, which returns a given post-collisional pair after interaction, is not guaranteed, unless we suppose that the collisional rule is a diffeomorphism of $\bar{\Omega}^2$ onto itself. Unfortunately, this assumption is not easy to satisfy in general.

In order to overcome this difficulty, we use a weak form of our problem, in the variable $x$ only, which seems a natural framework for such collision rules, as in [6]. The weak form of the collision kernels are presented below, where $\varphi = \varphi(x) \in C^0(\bar{\Omega})$ is a test function.

Note that, in both models studied below, the parameters are chosen in such a way that the Jacobians of the collisional mechanisms are always nonzero. These assumptions ensure the microreversibility of the collisions, which is a standard assumption of the kinetic theory.

**Conciliatory-conciliatory interactions.** Let $x$, $x_* \in \bar{\Omega}$ the pre-interaction opinions of two conciliatory agents, and $x^Q, x_*^Q \in \bar{\Omega}$ the opinions after interaction.

We denote by $Q(f,f)$ the associated kernel, which is defined by

$$\langle Q(f,f), \varphi\rangle = \beta_Q \iint_{\Omega^2} f(t,x) f(t,x_*) \left[\varphi\left(x^Q\right) - \varphi(x)\right] \, \mathrm{d}x \, \mathrm{d}x_*, \tag{1}$$

and $Q^+(f,f)$ the gain part of $Q(f,f)$, namely

$$\langle Q^+(f,f), \varphi\rangle = \beta_Q \iint_{\Omega^2} f(t,x) f(t,x_*) \varphi\left(x^Q\right) \, \mathrm{d}x \, \mathrm{d}x_*.$$

**Conciliatory-contradictory interactions.** Let $x$, $x_* \in \bar{\Omega}$ the respective pre-interaction opinions of a conciliatory agent and a contradictory one before an interaction, and $x^R, x_*^R \in \bar{\Omega}$ the opinions after interaction.

We denote by $R_1(f,g)$ and $R_2(f,g)$ the associated kernels, which respectively contribute to the time evolution of $f$ and of $g$. They are defined by

$$\langle R_1(f,g), \varphi\rangle = \beta_R \iint_{\Omega^2} f(t,x) g(t,x_*) \left[\varphi\left(x^R\right) - \varphi(x)\right] \, \mathrm{d}x \, \mathrm{d}x_*, \tag{2}$$

$$\langle R_2(f,g), \varphi\rangle = \beta_R \iint_{\Omega^2} f(t,x) g(t,x_*) \left[\varphi\left(x_*^R\right) - \varphi(x)\right] \, \mathrm{d}x \, \mathrm{d}x_*. \tag{3}$$

Their gain parts are respectively $R_1^+(f,g)$ and $R_2^+(f,g)$, i.e.

$$\begin{aligned}
\langle R_1^+(f,g), \varphi\rangle &= \beta_R \iint_{\Omega^2} f(t,x) g(t,x_*) \varphi\left(x^R\right) \, \mathrm{d}x \, \mathrm{d}x_*, \\
\langle R_2^+(f,g), \varphi\rangle &= \beta_R \iint_{\Omega^2} f(t,x) g(t,x_*) \varphi\left(x_*^R\right) \, \mathrm{d}x \, \mathrm{d}x_*.
\end{aligned}$$

**Contradictory-contradictory interactions.** Let $x$, $x_* \in \bar{\Omega}$ the pre-interaction opinions of two contradictory individuals, and $x^S, x_*^S \in \bar{\Omega}$ the post-interaction ones.

We eventually define the associated kernel $S(g,g)$ by

$$\langle S(g,g), \varphi\rangle = \beta_S \iint_{\Omega^2} g(t,x) g(t,x_*) \left[\varphi\left(x_*^S\right) - \varphi(x)\right] \, \mathrm{d}x \, \mathrm{d}x_*. \tag{4}$$

The associated gain term is denoted by $S^+(g,g)$, and defined by

$$\langle S^+(g,g),\varphi\rangle = \beta_S \iint_{\Omega^2} g(t,x)g(t,x_*)\varphi\left(x_*^S\right)\, \mathrm{d}x\, \mathrm{d}x_*.$$

**Analytical form of the model.** In the collisional terms, the parameters $\beta_Q$, $\beta_R$ and $\beta_S$ govern the probability that the associated interaction can occur. In our model, we choose them as constants. This is the simplest possible assumption, which means that the probability of interaction of two individuals does not depend on their respective opinions. Of course, other choices, based on sociological considerations, are possible.

Note that, in (1)–(4), the post-interaction opinions $x^Q$, $x_*^Q$, $x^R$, $x_*^R$, $x^S$ and $x_*^S$ only appear as an argument of the test-function $\varphi$. It is also clear that the collision operators only act on the opinion variable. Moreover, if we suppose that $f(t,\cdot)$ and $g(t,\cdot)$ lie in $L^1(\Omega)$, then the gain and loss parts of the operators and, consequently, the operators themselves, also lie in $L^1(\Omega)$ for any $t$.

Let $T > 0$. The evolution of $f$ and $g$ is given by the following system of integro-differential equations, in the weak sense in the variable $x$, where both $f^{\mathrm{in}}$ and $g^{\mathrm{in}}$ are $L^1(\Omega)$ and nonnegative,

$$\partial_t f = Q(f,f) + R_1(f,g), \tag{5}$$

$$\partial_t g = R_2(f,g) + S(g,g), \tag{6}$$

with initial conditions

$$f(0,x) = f^{\mathrm{in}}(x), \qquad g(0,x) = g^{\mathrm{in}}(x) \qquad \text{a.e. } x\in\Omega. \tag{7}$$

Equations (5)–(6) are defined for almost every $t \in [0,T]$ and for test-functions $\varphi \in C^0(\bar\Omega)$. We do not need any boundary condition because the collision rules are chosen to prevent the interactions from destroying the bounds of the interval $\bar\Omega$.

## 3. Mathematical properties

This section is devoted to state and study some mathematical properties of (5)–(7), which do not depend on the collision rules. We first obtain some a priori estimates and then deduce a result which ensures the existence of weak solutions to (5)–(7). The proofs of the first results of this section are quite straightforward, and we refer to [6] for more details.

**Proposition 1.** *Let $(f,g)$ be a nonnegative weak solution to* (5)–(7) *with nonnegative initial data $f^{\mathrm{in}}$, $g^{\mathrm{in}} \in L^1(\Omega)$. Their respective total masses are conserved, i.e. for almost every $t \in [0,T]$,*

$$\|f(t,\cdot)\|_{L^1(\Omega)} = \left\|f^{\mathrm{in}}\right\|_{L^1(\Omega)}, \qquad \|g(t,\cdot)\|_{L^1(\Omega)} = \left\|g^{\mathrm{in}}\right\|_{L^1(\Omega)}.$$

Proposition 1 means that the number of agents in each group of the population is conserved. This property is not realistic if we consider long-time forecasts. Indeed, in such situations, we should also consider processes of birth and death, which also lead to oscillations in the total number of individuals. But usually, as in the case of elections or referenda, the interest of such models is to deduce short-term forecast by using, as an initial datum, poll results. The quantities of interest are then the *macroscopic observables*

$$\int_A f(t,x)\,\mathrm{d}x, \ \int_A g(t,x)\,\mathrm{d}x,$$

where $A \subseteq \Omega$. Typically, $A = (-1,-\varepsilon)$ or $A = (\varepsilon,1)$ for some $\varepsilon \geq 0$ and the integrals represent the number (density) of individuals with opinion belonging to the set $A$.

Since $|x| \leq 1$, from the mass conservation, we immediately deduce that all the moments of both $f$ and $g$ are bounded.

**Corollary 1.** *Let $(f,g)$ be a nonnegative weak solution to* (5)–(7) *with nonnegative initial data $f^{\mathrm{in}}$, $g^{\mathrm{in}} \in L^1(\Omega)$. Then we have, for almost every $t \in [0,T]$ and $n \geq 1$,*

$$\left|\int_\Omega x^n f(t,x)\,\mathrm{d}x\right| \leq \|f^{\mathrm{in}}\|_{L^1(\Omega)}, \qquad \left|\int_\Omega x^n g(t,x)\,\mathrm{d}x\right| \leq \|g^{\mathrm{in}}\|_{L^1(\Omega)}.$$

In order to prove the existence of weak solutions to (5)–(7), we first need the following proposition, the proof of which mainly lies on the Cauchy-Lipschitz theorem and the Duhamel formula.

**Proposition 2.** *Let $\mu_1$, $\mu_2$ be nonnegative constants, $\sigma_1$, $\sigma_2$ nonnegative functions in $C^0([0,T];L^1(\Omega))$, and $u^{\mathrm{in}}$, $v^{\mathrm{in}}$ nonnegative initial data in $L^1(\Omega)$. The system*

$$\partial_t u + \mu_1\ u = \sigma_1, \qquad \partial_t v + \mu_2\ v = \sigma_2, \tag{8}$$

*with initial conditions*

$$u(0,\cdot) = u^{\mathrm{in}}, \qquad v(0,\cdot) = v^{\mathrm{in}}, \tag{9}$$

*has a unique solution $(u,v) \in \big(C^0([0,T];L^1(\Omega))\big)^2$. Moreover, both $u$ and $v$ are nonnegative.*

Thanks to the previous result, we can now prove the following existence theorem.

**Theorem 1.** *Let $f^{\mathrm{in}}$, $g^{\mathrm{in}}$ be nonnegative functions in $L^1(\Omega)$. Then there exists $(f,g) \in L^\infty(0,T;L^1(\Omega)) \times L^\infty(0,T;L^1(\Omega))$ which solves* (5)–(6) *with initial conditions* (7)*, where the equations take sense in the distributional sense on $(-T,T)$.*

*Proof.* The proof is quite similar to the one of the main result in [6]. Let us set

$$\varrho_f = \int_\Omega f^{\mathrm{in}}(x_*)\ \mathrm{d}x_* \geq 0, \qquad \varrho_g = \int_\Omega g^{\mathrm{in}}(x)\ \mathrm{d}x \geq 0.$$

We consider the sequence $(f^n, g^n)_{n\in\mathbb{N}}$ inductively defined by $f^0 = 0$, $g^0 = 0$, and, for $n \geq 1$, as weak solutions of

$$\partial_t f^{n+1} + (\beta_Q \varrho_f + \beta_R \varrho_g) f^{n+1} = Q^+(f^n, f^n) + R_1^+(f^n, g^n), \tag{10}$$

$$\partial_t g^{n+1} + (\beta_R \varrho_f + \beta_S \varrho_g) g^{n+1} = R_2^+(f^n, g^n) + S^+(g^n, g^n), \tag{11}$$

altogether with the initial conditions $f^n(0,\cdot) = f^{\mathrm{in}}$ and $g^n(0,\cdot) = g^{\mathrm{in}}$.

With a constant test function equal to 1, it is clear that, for all $n \in \mathbb{N}$, we have

$$\int_\Omega f^n \,\mathrm{d}x \leq \varrho_f, \qquad \int_\Omega g^n \,\mathrm{d}x \leq \varrho_g.$$

The existence of $f^n$ and $g^n$ in $C^0([0,T]; L^1(\Omega))$ as nonnegative solutions of (10)–(11) is obtained by induction thanks to Proposition 2, remembering that $Q^+(f^n, f^n)$, ${R_1}^+(f^n, g^n)$, ${R_2}^+(f^n, g^n)$ and $S^+(g^n, g^n)$ all belong to $C^0([0,T]; L^1(\Omega))$.

We can prove, by induction again, that $(f^n)$ and $(g^n)$ are non decreasing sequences. Therefore, by monotone convergence, there exist $f$, $g \in L^\infty(0,T; L^1(\Omega))$, such that $(f^n)$ and $(g^n)$ converge to $f$ and $g$, almost everywhere and in $L^\infty(0,T; L^1(\Omega))$.

We still have to prove that $(f,g)$ satisfies the initial conditions (7), which is quite clear, and solves (5)–(6) in a distributional sense. Let us choose a test function $\varphi \in C^0(\bar{\Omega})$, and a test function $\psi \in C_0^\infty([-T;T])$ (compact-supported $C^\infty$ functions). Equations (5)–(6) can be written in a weak form, using these test functions. We investigate what happens when $n \to +\infty$ in this formulation.

First, the time derivatives and initial data do not induce any difficulty, when $n \to +\infty$. To treat the linear term with the indices $n+1$, we only have to use Proposition 1 to obtain the loss terms of the collision kernels. Eventually, we have to deal with the nonlinear terms involving $f^n$ and $g^n$. For instance, if we set $M = \sup\{|\varphi(x)|;\ x \in \bar{\Omega}\}$, we can note that

$$\iint_{\Omega^2} |f^n(t,x) g^n(t,x_*) - f(t,x) g(t,x_*)| \; |\varphi(x')| \,\mathrm{d}x\,\mathrm{d}x_*$$
$$\leq \varrho_g M \left\| f^n(t,\cdot) - f(t,\cdot) \right\|_{L^1(\Omega)} + \varrho_f M \left\| g^n(t,\cdot) - g(t,\cdot) \right\|_{L^1(\Omega)},$$

which goes to 0 when $n \to +\infty$. Using this argument or a similar one, it is easy to recover the gain terms of the collision kernels.

We are then able to let $n$ go to $+\infty$ in the weak formulation of (5)–(6) and obtain the required result. □

## 4. The collision rules

This section is devoted to describe, in a precise way, the psychological dynamics that we take into account. For each individual, the exchange of opinions with another member of the population is represented, in the model, by a collisional rule that quantifies the modifications in the opinions originated by the exchange itself.

In what follows, $(x, x_*) \in \bar{\Omega}^2$ are the pre-interaction opinions of the agents, whereas $(x', x'_*) \in \bar{\Omega}^2$ represent their opinions after the discussion, where the primes denote the types of variables $Q$, $R$ and $S$.

As we shall see, the choice of the collision rules is crucial and heavily conditions the time evolution of the distribution functions given by Equations (5)–(7).

We here propose two models, based on two different psychological mechanisms for contradictory individuals, which allow to describe two types of collective behaviour observed in real situations. Both psychological rules translate the idea that the contradictory way of thinking tends to oppose the effects of the consensus rule.

The comparison between these two models will allow to investigate how the way of thinking of a small fraction of the population can dramatically influence the time evolution of the system, and to understand the power of minorities.

An essential ingredient of both collision rules is the *attraction function* $\eta$, a smooth function which describes the degree of attraction of the average opinion with respect to the starting opinion of the individual. In the first model, we introduce also the *reaction function* $\alpha$, a smooth function which modulates the reaction of a contradictory individual during the exchange process.

In order to make the models unaffected by the change of label of the two extreme opinions, we shall always assume that both $\eta$ and $\alpha$ are even. Moreover, we suppose that $\eta : \bar{\Omega} \to \mathbb{R}$ and $\alpha : \bar{\Omega} \to \mathbb{R}$ are $C^1$, and such that $0 \leq \eta < 1$, $0 < \alpha \leq 1$. Consequently, the interactions do not destroy the bounds of the interval $\bar{\Omega}$.

In order to translate the idea, well accepted in the literature [6, 19, 12, 10], that extreme opinions are more stable than moderate ones for conciliatory individuals, we suppose that $\eta'(x) \geq 0$ when $x \geq 0$. We also assume that the attraction and reaction functions are such that the Jacobians of the collisional mechanisms are always non zero.

We respectively name our two models the *twist* and *swing* models. This choice of nomenclature will be clear by observing the numerical results of Section 5. We keep the notations defined in Section 2 for the post-interaction opinions in each case. In both models, a conciliatory individual behaves in the same way when interacting with another conciliatory agent or a contradictory person. The behaviour of contradictory people differs in the two models, as described below.

4.1. **Twist model.** In the following paragraphs, we detail the involved collision rules. Each contradictory individual thinks the opposite of what he should have thought if he was a conciliatory one, and the more he has a strong pre-interaction opinion, the more he changes his mind.

4.1.1. *Exchange of opinions between two conciliatory individuals.* Let $x$, $x_* \in \bar\Omega$ denote the opinions of two conciliatory agents before an interaction. The interaction is described by the rule defined in [6]: the stronger opinions are less attracted towards the average than the weaker ones. The mechanism which returns the post-interaction opinions $x^Q$, $x_*^Q$ is given by

$$x^Q = \frac{x+x_*}{2} + \eta(x)\frac{x-x_*}{2}, \tag{12}$$

$$x_*^Q = \frac{x_*+x}{2} + \eta(x_*)\frac{x_*-x}{2}. \tag{13}$$

4.1.2. *Exchange of opinions between a conciliatory individual and a contradictory one.* Let $x$, $x_* \in \bar\Omega$ the respective opinions of a conciliatory agent and a contradictory one before an interaction. Whereas the conciliatory individual still follows the consensus rule (12), the post-interaction contradictory opinion is computed using the value given by (13), and then somehow taking the opposite value, to model the contradictory effect. The collision rule that individuates the post-interaction opinions $x^R$, $x_*^R$ writes

$$x^R = \frac{x+x_*}{2} + \eta(x)\frac{x-x_*}{2}, \tag{14}$$

$$x_*^R = -\alpha(x_*)\left[\frac{x_*+x}{2} + \eta(x_*)\frac{x_*-x}{2}\right]. \tag{15}$$

4.1.3. *Exchange of opinions between two contradictory individuals.* Let $x$, $x_* \in \bar\Omega$ the opinions of the two contradictory agents before an interaction. The opinion exchange gives a pair of post-interaction opinion variables $x^S$, $x_*^S$, which are given by

$$x^S = -\alpha(x)\left[\frac{x+x_*}{2} + \eta(x)\frac{x-x_*}{2}\right], \tag{16}$$

$$x_*^S = -\alpha(x_*)\left[\frac{x_*+x}{2} + \eta(x_*)\frac{x_*-x}{2}\right]. \tag{17}$$

4.2. **Swing model.** This model substantially differs from the previous one because of the psychological behaviour of contradictory individuals. Indeed, each contradictory agent opposes himself to his interlocutor by reinforcing his pre-interaction opinion, and the more he has a strong pre-interaction opinion, the less he changes his mind.

4.2.1. *Exchange of opinions between two conciliatory individuals.* The interaction between two conciliatory individuals is still defined by (12)–(13).

4.2.2. *Exchange of opinions between a conciliatory individual and a contradictory one.* Let $x$, $x_* \in \bar\Omega$ the respective opinions of a conciliatory agent and a contradictory one before an interaction. After interaction, the contradictory individual tends to oppose the post-interaction opinion of his peer, i.e. the post-interaction opinions $x^R$, $x_*^R$ are defined by

$$x^R = \frac{x+x_*}{2} + \eta(x)\frac{x-x_*}{2}, \tag{18}$$

$$x_*^R = \begin{cases} 1 - \dfrac{(1-x^R)(1-x_*)}{(1-x)} & \text{if } x < x_*, \\ x_*(= x = x^R) & \text{if } x = x^R, \\ \dfrac{(1+x^R)(1+x_*)}{(1+x)} - 1 & \text{if } x > x_*. \end{cases} \tag{19}$$

We note that the post-interaction opinion $x_*^R$ is well defined because the range of the validity of the formulae prevents the denominators from vanishing. The interaction clearly does not destroy the bounds of $\bar{\Omega}$. Finally, the continuity of the mechanism with respect to $(x, x_*)$ is also ensured when $x = x_*$. The effect of this rule with respect to the post-interaction opinion of the contradictory individual is the following: if his pre-interaction opinion is less than the pre-interaction opinion of his peer, the corresponding post-interaction opinion is greater than the post-interaction opinion of his partner. The opposite situation happens when the order of the opinion before an interaction is inverted. Figure 1 may help the reader to understand the mechanism, using the intercept theorem.

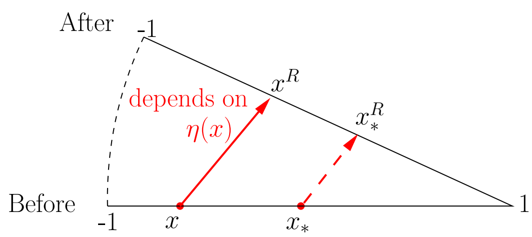


FIGURE 1. Principle of collision rules (18)–(19)

4.2.3. *Exchange of opinions between two contradictory individuals.* Let $x$, $x_* \in \bar{\Omega}$ the pre-interaction opinions of two contradictory agents. The interaction leading to the post-interaction opinions $x^S$, $x_*^S$ is given by

$$x^S = \begin{cases} \dfrac{x-1}{2} + \eta(x)\dfrac{x+1}{2} & \text{if } x \le x_*, \\ \dfrac{x+1}{2} + \eta(x)\dfrac{x-1}{2} & \text{if } x > x_*, \end{cases} \tag{20}$$

$$x_*^S = \begin{cases} \dfrac{x_*+1}{2} + \eta(x_*)\dfrac{x_*-1}{2} & \text{if } x \le x_*, \\ \dfrac{x_*-1}{2} + \eta(x_*)\dfrac{x_*+1}{2} & \text{if } x > x_*. \end{cases} \tag{21}$$

4.3. **About the collision mechanisms.** The sets of attraction and reaction functions are not empty. Indeed, a possible choice for functions $\eta$ and $\alpha$ is $\eta(x) = H$ and $\alpha(x) = A$ for $x \in \bar{\Omega}$, where the constants $H$ and $A$ both satisfy $0 < H, A < 1$. Easy but tedious computations finally ensure that, in that case, the Jacobians of all the collisions rules (12)–(21) are defined and nonzero.

In the so-called swing model, we had to define a specific behaviour for contradictory agents. For instance, (19) could have been really close to (21), but we thought that the dependence of $x_*^R$ on $x$ would not have been strong enough. Consequently, we chose (19) quite different from (21), but satisfies the required mathematical properties: significant dependence on $x$, smoothness of the rule with respect to both variables $x$ and $x_*$, and preserving the bounds of $\bar{\Omega}$ thanks to the intercept theorem. In terms of sociophysical meaning, (19) is just a way to write that, with the contradictory behaviour, $x$ and $x_*$ get closer to $\pm 1$, while becoming $x^R$ and $x_*^R$, as in Fig. 1.

## 5. Numerical tests

In this section, we present some numerical results concerning both models presented above. The computations were performed by using a numerical code written in C. We consider a regular subdivision $(x_0, \cdots, x_N)$ of $\Omega$, with $N \geq 1$. The functions $f$ and $g$ are computed at the center $x_{i+1/2}$ of each interval $[x_i, x_{i+1}]$, $0 \leq i \leq N-1$, and we choose $N = 1000$. Other values of $N$ have also been tested, without any significant changes in the numerical results.

Since $\|f^{\text{in}}\|_{L^1(\Omega)}$ and $\|g^{\text{in}}\|_{L^1(\Omega)}$ may not have the same order of magnitude, we respectively replace $f$ and $g$ by

$$\tilde{f} := \frac{f}{\|f^{\text{in}}\|_{L^1(\Omega)}}, \qquad \text{and} \qquad \tilde{g} := \frac{g}{\|g^{\text{in}}\|_{L^1(\Omega)}}.$$

Then $\tilde{f}$ and $\tilde{g}$ solve the same kind of equations (5)–(6) as $f$ and $g$, but $\beta_Q$ becomes $\beta_Q \|f^{\text{in}}\|_{L^1(\Omega)}$ and $\beta_S$ is replaced by $\beta_S \|g^{\text{in}}\|_{L^1(\Omega)}$. The case of $R$ is more intricate: we need to use two parameters $\beta_{R_1} := \beta_R \|g^{\text{in}}\|_{L^1(\Omega)}$ for $R_1$ and $\beta_{R_2} := \beta_R \|f^{\text{in}}\|_{L^1(\Omega)}$ for $R_2$. Note that, from the numerical viewpoint, that means that (5) and (6) are now somehow uncoupled. On the other hand, both $\|\tilde{f}(t,\cdot)\|_{L^1(\Omega)}$ and $\|\tilde{g}(t,\cdot)\|_{L^1(\Omega)}$ remain constant and equal to 1, for every $t$. Note that implies that the integrals of type $I^{\pm}$ are then numerically well defined.

The scheme is time-split into four parts composed of the four types of collisions described below. To numerically perform the collisions, we use a slightly modified Bird method [4], as in [6], including subcycling when needed. More precisely, at each time step, the population is individuated into $n(t)$ numerical agents, with $n_i(t)$ agents of opinion $x_i$. Each $n_i(t)$ are computed in the following way. At initial time, we choose the fraction of the population $\omega$ represented by one numerical agent during the whole computation. The quantity $n_i(t)$ is obtained as the upper integer part of $f(t,x_i)/\omega$. Then, during each time step, we perform $\gamma = \beta_Q/\beta_S$ collisions of type $Q$, $\delta_1 = \beta_{R_1}/\beta_S$ collisions of type $R_1$, $\delta_2 = \beta_{R_2}/\beta_S$ collisions of type $R_2$ and one collision of type $S$. For the sake of simplicity, the parameters of the computation are chosen such that $\gamma$, $\delta_1$ and $\delta_2$ are integers. Of course, all the collisions of any type are performed in a random order. Note that our scheme prevents the opinions from going out of $\bar{\Omega}$, and conserves the population of each group.

In what follows, we are mainly interested in the computation of the quantities

$$I^-(f+g) = \int_{-1}^{0} (f+g)(t,x)\,\mathrm{d}x, \quad I^+(f+g) = \int_{0}^{1} (f+g)(t,x)\,\mathrm{d}x.$$

When normalized to $\|f^{\mathrm{in}} + g^{\mathrm{in}}\|_{L^1(\Omega)}$, they can be seen as the fraction of agents who respectively favour negative and positive opinions. Furthermore, we can sometimes assume that contradictory individuals do not wish to vote (or be part of a poll), and consequently may also be interested in the quantity

$$I^+(f) = \int_0^1 f(t,x)\,\mathrm{d}x.$$

In each numerical tests, the collision frequencies are set to $\beta_S = 1$, $\beta_R = 2$, $\beta_Q = 5$. That means that we always consider that the interactions involving contradictory people are less frequent that the ones involving conciliatory individuals. Besides, the initial data are chosen such that $\|f^{\mathrm{in}}\|_{L^1(\Omega)} = 1$ and $\|g^{\mathrm{in}}\|_{L^1(\Omega)} = 0.1$, so that conciliatory individuals are majority, but contradictory people are a significant part of the whole population. We also tried smaller values of $\|g^{\mathrm{in}}\|_{L^1(\Omega)}$. The same kind of behaviours shown below are recovered, but the time scales get smaller and the interesting transient effects cannot really be pointed out.

Eventually, we choose, for $x \in \bar{\Omega}$, $\eta(x) = (1+x^2)/4$, and $\alpha(x) = (1+x^2)/2$ when required. It is quite difficult to find a precise sociological meaning for those choices. We refer to [6] for the discussion about the choice of $\eta$. Nevertheless, both functions satisfy the required assumptions, and we emphasize that we have numerically checked that all the Jacobians were nonzero.

### 5.1. **Twist model.**

5.1.1. *Uniform contradictory group within an uncentred conciliatory population.* In this first test, we show the behaviour of the model with an Heaviside-step-like initial datum for conciliatory individuals, and a constant initial datum for contradictory people:

$$f^{\mathrm{in}}(x) = \begin{cases} 2 & \text{if } x < -0.5, \\ 0 & \text{if } x > -0.5, \end{cases} \qquad g^{\mathrm{in}}(x) = 0.05. \tag{22}$$

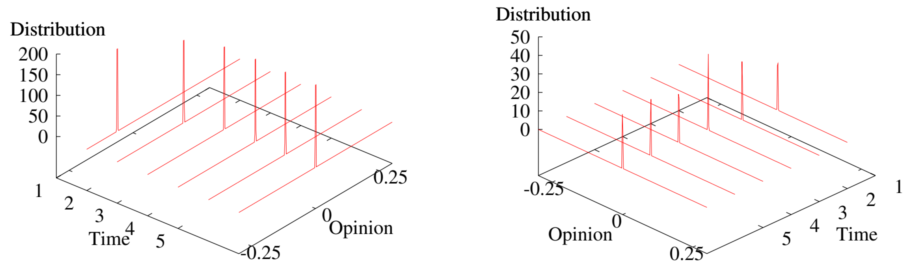


FIGURE 2. Twist model: graphs of (a) $f$ and (b) $g$, with initial data (22).

In Figure 2, we plot the time and opinion evolution of the distribution functions. We focus on $-0.3 \leq x \leq 0.3$ and $1 \leq t \leq 6$ because $f$ and $g$ quickly reach their equilibrium shapes (Dirac masses mostly centred at 0). But small changes of the support set imply, in the transient period, significant changes on $I^+(f+g)$, see Figure 3.

As expected, the initial conditions for both groups do not have any influence after a transient time. The distribution functions $f$ and $g$ asymptotically get a Dirac-mass shape. Figure 3 ensures that the Dirac mass centres for both $f$ and $g$ go to 0, since $I^+(f+g)$ goes to $\|f^{\text{in}} + g^{\text{in}}\|_{L^1(\Omega)}/2 = 0.55$. The convergence to the equilibrium for $g$ is slower than the one for $f$ (even it is not obvious in Figure 2.

The twist model then clearly results in a centred population with no precise opinion, since both Dirac masses are asymptotically centred at 0. It is interesting to note that, in the contradictory-free diffusionless case described in [6], when $g$ does not appear, $f$ also converges to a Dirac mass, but its centre is given by the average opinion of the initial conciliatory population.

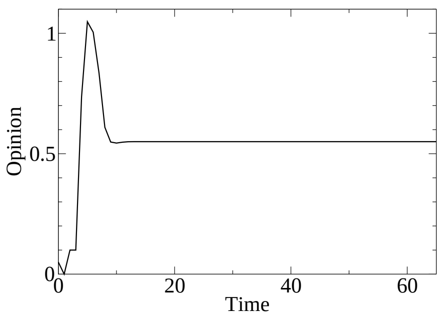


FIGURE 3. Twist model: graph of $I^+(f+g)$ with initial data (22).

5.1.2. *Disjoint-supported initial data.* Let us now investigate the case when the support sets of the initial data are disjoint. Indeed, in that situation, the support sets of $f$ and $g$ may still remain disjoint because of the collision rules involving contradictory people. More precisely, we consider the following initial data:

$$f^{\text{in}}(x) = \begin{cases} 2 & \text{if } x < -0.5, \\ 0 & \text{if } x > -0.5, \end{cases} \quad \text{and} \quad g^{\text{in}}(x) = \begin{cases} 0 & \text{if } x < 0.5, \\ 0.2 & \text{if } x > 0.5. \end{cases} \tag{23}$$

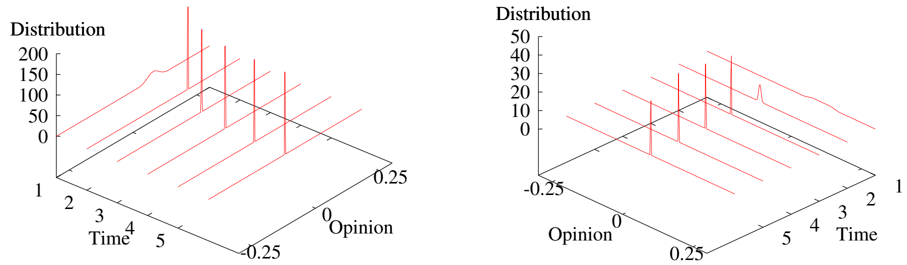


FIGURE 4. Twist model: graphs of (a) $f$ and (b) $g$, with initial data (23).

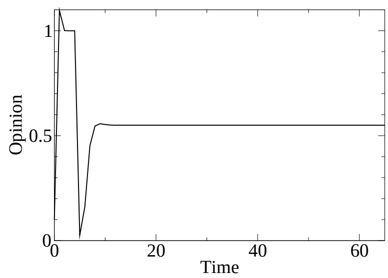


FIGURE 5. Twist model: graph of $I^+(f+g)$, with initial data (23).

Again, both functions $f$ and $g$ fastly converge towards Dirac masses centred at 0, and the conclusion we obtained in 5.1.1 seems to hold.

However, for small time, each conciliatory individual has a positive opinion and each contradictory individual a negative one (see Figure 4), a situation which is significantly different with respect to the initial data. After a transient period, where a *double* reversal of the majority takes place (see Figure 5), the system eventually reaches an equilibrium configuration.

As a conclusion, the twist model forecasts situations which lead to a fifty/fifty result. The phenomenon of "hung elections" [13] is the natural issue of this model. But one of the main interest of our model is its behaviour during a transient time, with unexpected results due to contradictory individuals.

### 5.2. **Swing model.**

5.2.1. *Uniform contradictory group within an uncentred conciliatory population.* We start with the first same numerical test as in 5.1.1, i.e. with initial data given by (22).

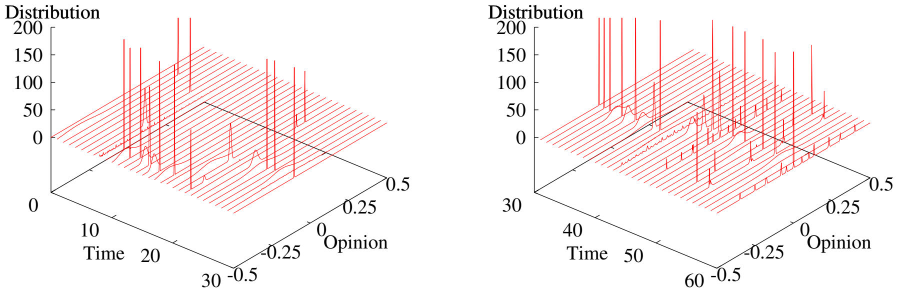


FIGURE 6. Swing model: graphs of $f$, with initial data (22).

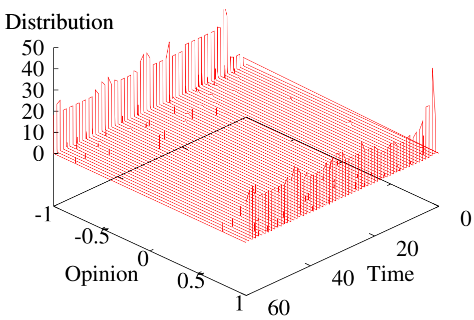


FIGURE 7. Swing model: graph of $g$, with initial data (22).

As we can see in Figures 6–7, the distribution function $f$ looks either like a unique peak, or like a sum of several peaks, or even like a Gaussian-like function (*e.g.* at time $t = 17$) . Its centre seems to randomly swing around 0. Thus, for instance, at $t = 2$ or at $t = 46$, each conciliatory individual has a positive opinion, and at $t = 30$, each one of them has a negative opinion. This behaviour is of course linked to the one of $g$, which seems to converge towards the sum of two peaks, centred at $\pm 1$. But in fact, the peaks' heights do not remain constant with respect to $t$ and, sometimes, we can find secondary peaks in addition to the main ones.

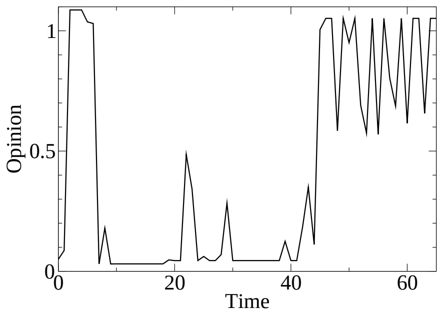


FIGURE 8. Swing model: graph of $I^+(f+g)$, with initial data (22).

Those behaviours are very different from the results given by the twist model. In Figure 8, we note that there are three majority changes in the opinion for $t < 60$.

This time, we may not get a fifty/fifty situation, and no tendency comes up: when time grows, $I^+(f+g)$ can randomly go below or above $\|f^{\mathrm{in}} + g^{\mathrm{in}}\|_{L^1(\Omega)}/2 = 0.55$, and there is no asymptotic equilibrium.

5.2.2. *Disjoint-supported initial data.* Let us now study the behaviour of the swing model with the set of initial data (23).

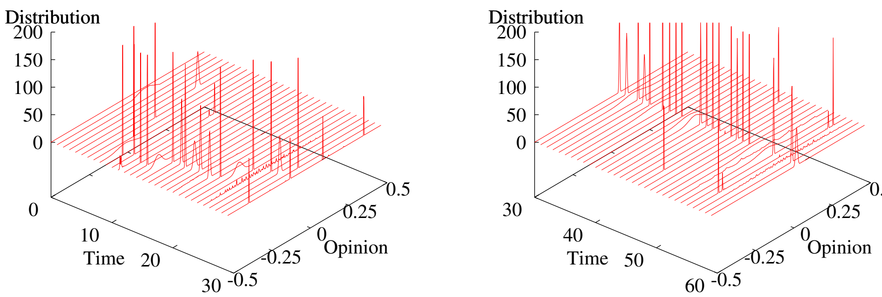


FIGURE 9. Swing model: graphs of $f$, with initial data (23).

In Figure 9, the distribution function $f$ has the same shape as in Figure 6. A striking fact is that, at the first time step, each conciliatory individual has already a positive opinion: the first reversal occurs almost instantaneously.

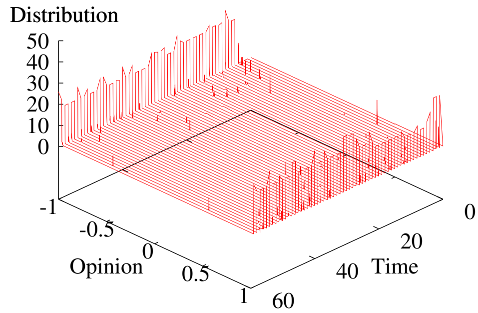


FIGURE 10. Swing model: graph of $g$, with initial data (23).

On the other hand, on Figure 10, as on Figure 7, $g$ seems to converge again towards the sum of two peaks centred at $\pm 1$, with possible secondary peaks.

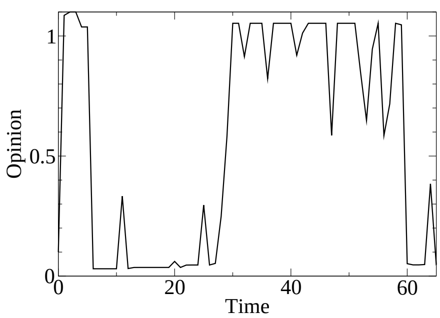


FIGURE 11. Swing model: graph of $I^+(f+g)$, with initial data (23).

Eventually, in Figure 11, we note that there are four changes of majority in the opinion at small time ($t < 60$).

5.2.3. *Large-time behaviour.*

Figure 12 shows clearly that, in the swing model, we may obtain no asymptotic equilibrium at all, since majority changes can occur. The integral $I^+(f)$, which is the quantity plotted in Figure 12, visualizes the great influence of the contradictory portion of the population on the behaviour of the conciliatory individuals.

Anyway, that means that there exist psychological behaviours which can lead to unpredictable election results, caused by unstable majorities.

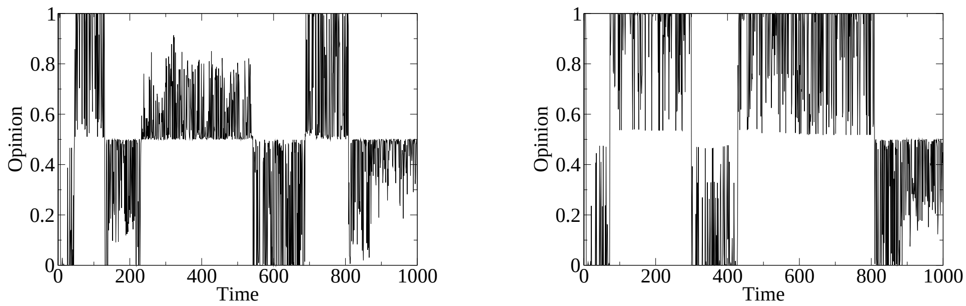


FIGURE 12. Swing model: graphs of $I^+(f)$, with initial data (a) (22) and (b) (23).

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L.B.: UPMC Univ Paris 06, UMR 7598 LJLL, Paris, F-75005, France & INRIA Paris-Rocquencourt, REO Project team, BP 105, F-78153 Le Chesnay Cedex, France
*E-mail address*: laurent.boudin@upmc.fr

A.M.: UPMC Univ Paris 06, UMR 7598 LJLL, Paris, F-75005 France
*E-mail address*: mercier.aurore@gmail.com

F.S.: INRIA Paris-Rocquencourt, REO Project team, BP 105, F-78153 Le Chesnay Cedex, France & Dipartimento di Matematica, Università degli Studi di Pavia, Via Ferrata 1 - I-27100 Pavia, Italy
*E-mail address*: francesco.salvarani@unipv.it