ASYMPTOTIC BEHAVIOR OF DEGENERATE LINEAR TRANSPORT EQUATIONS

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ABSTRACT. We study in this paper a few simple examples of hypocoercive systems in which the coercive part is degenerate. We prove that the (completely explicit) speed of convergence is at least of inverse power type (the power depending on the features of the considered system).

1. INTRODUCTION

We consider non-homogeneous (in space) transport equations of the type

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \sigma(x)\,(\bar{f} - f), \tag{1}$$

where $f := f(t,x,v)$ is the density of particles which at time t and point x move with velocity v. Here $\bar{f}(t,x) = \int_V f(t,x,v)\,dv$, where V is a bounded set (of $\mathbb{R}^d$) of velocities of measure 1. The right-hand side of Equation (1) describes a process of isotropization of the velocities of the particles. This process has an intensity $\sigma(x)$ which is not necessarily bounded below by a strictly positive constant (in the vocabulary of radiative transfer, this would correspond to regions which are completely transparent).

For the sake of simplicity, we shall systematically consider that the solutions are periodic (of period 1) in all components of x, that is $x \in \mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$, and nonnegative times $t \in \mathbb{R}_+$.

We finally introduce initial data

$$f(0,x,v) = f_0(x,v). \tag{2}$$

We shall also consider a simplified one-dimensional model of (1), in which the velocities are $v = \pm 1$. This is a variant of the well-known Goldstein-Taylor model [6], [11], which describes the behavior of a gas composed of two kinds of particles moving parallel to the x-axis with constant speeds, of equal modulus $c = 1$, one in the positive x-direction with density $u(x,t)$, the other in the negative x-direction with density $v(x,t)$.

The corresponding system of equations is:

$$
\left\{
\begin{aligned}
&\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = \sigma(x)(v-u)\\
&\frac{\partial v}{\partial t} - \frac{\partial v}{\partial x} = \sigma(x)(u-v),
\end{aligned}
\right.
\tag{3}
$$

where $u := u(t,x)$, $v := v(t,x)$, $x \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$, $t \geq 0$.

Such set of equations will satisfy the initial conditions

$$
u(0,x) = u_0(x), \qquad v(0,x) = v_0(x). \tag{4}
$$

If σ were bounded from below by a strictly positive constant, then a variant of the strategy proposed by Mouhot and Neumann in [9] would lead to prove the exponential decay (with explicit rates) in time of the solutions of Equation (1) or System (3) towards the unique equilibrium state of the system.

However, this result has no obvious extension in the case of a vanishing cross section (even if such a degeneracy happens at only one point). A reasonable conjecture is that when the equilibrium is still unique, then some explicit (non necessarily exponential) rate should still exist.

The goal of this paper is to prove this property under reasonable assumptions on the cross section. More precisely, we shall suppose that it satisfies the properties given in the following assumption:

Assumption 1: Let $\sigma : \mathbb{T}^d \to \mathbb{R}_+$ be a function satisfying the following property: there exist $x_i \in \mathbb{T}^d$, $i = 1, \dots, N$, $C_\sigma > 0$ and $\lambda_\sigma > 0$ such that

$$
\text{for a.e. } x \in \mathbb{T}^d, \qquad \sigma(x) \geq C_\sigma \inf_{i=1,\dots,N} |x - x_i|^{\lambda_\sigma}.
$$

Our results are summarized in the following theorems, which deal respectively with the transport equation and the Goldstein-Taylor model:

Theorem 1.1. *Consider the linear transport model* (1)-(2) *in the domain* $\mathbb{T}^d$ *(*$d \in \mathbb{N}$*) with a cross section* $\sigma \in L^\infty \cap H^1(\mathbb{T}^d)$ *satisfying Assumption* 1 *and* f_0 *such that* $f_0 \in L^\infty(\mathbb{T}^d \times V)$, $\nabla_x \bar{f}_0 \in L^2(\mathbb{T}^d)$, *and* $v \otimes v : \nabla_x \nabla_x f_0 \in L^2(\mathbb{T}^d \times V)$.

Then there exists a unique nonnegative solution $f := f(t,x,v)$ *to this system in* $C(\mathbb{R}_+; L^2(\mathbb{T} \times V))$, *which converges when* $t \to +\infty$ *to its asymptotic profile*

$$
f_\infty(x,v) := \int_{\mathbb{T}^d} \int_V f_0(y,w)\, dw dy.
$$

Moreover, the following estimate holds:

$$||f(t,\cdot,\cdot) - f_\infty||^2_{L^2(\mathbb{T}\times V)} \le C_1\, t^{-\frac{1}{1+2\,\lambda_\sigma}} \tag{5}$$

where C_1 is a constant depending on C_σ, λ_σ, $||\sigma||_{H^1(\mathbb{T})\cap L^\infty(\mathbb{T})}$, and f_0, which can be explicitly estimated in terms of those quantities.

Theorem 1.2. *Consider the generalized Goldstein-Taylor model* (3)-(4) *in the domain $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ with a cross section $\sigma \in H^1(\mathbb{T})$ satisfying Assumption* 1 *and (u_0, v_0) in $H^2(\mathbb{T}) \times H^2(\mathbb{T})$.*

Then there exists a unique nonnegative solution $(u, v) := (u(t, x), v(t, x))$ to this system in $C(\mathbb{R}_+; L^2(\mathbb{T}))^2$ which converges when $t \to +\infty$ to its asymptotic profile

$$(u_\infty, v_\infty) := \left(\frac{1}{2}\int (u_0 + v_0)\, dx, \frac{1}{2}\int (u_0 + v_0)\, dx\right).$$

Moreover, the following estimate holds:

$$||u(t,\cdot) - u_\infty||^2_{L^2} + ||v(t,\cdot) - v_\infty||^2_{L^2} \le C_2\, t^{-\frac{1}{1+\lambda_\sigma}}, \tag{6}$$

where C_2 is a constant depending on C_σ, λ_σ, $||\sigma||_{H^1(\mathbb{T})}$ and u_0, v_0, which can be explicitly estimated in terms of those quantities.

Finally, if the initial data (u_0, v_0) belong to $C^\infty(\mathbb{T}) \times C^\infty(\mathbb{T})$, and if the cross section σ also lies in $C^\infty(\mathbb{T})$, then estimate (6) *can be replaced by*

$$||u(t,\cdot) - u_\infty||^2_{L^2} + ||v(t,\cdot) - v_\infty||^2_{L^2} \le C_3\, t^{-\frac{3}{\lambda_\sigma}+\delta}, \tag{7}$$

for any $\delta > 0$. Here C_3 is a constant which now depends on C_σ, λ_σ, δ, $||\sigma||_{W^{k,\infty}}$ (for all $k \le k_0(\delta)$) and u_0, v_0, which can be explicitly estimated in terms of those quantities.

Note that it is not known if exponential (or even "almost exponential") convergence holds for these models. It is also not known if the method of hypocoercivity such as described (for example) in [9], [7], [12] can be used (though this seems likely). The proof presented here relies on the older method introduced in [1], [3] and [4]. The different rates of convergence obtained in Theorems 1.1 and 1.2 reflect the possibility to use interpolations which have a different power, depending on the a priori smoothness of the solution of the equations.

The ideas developed in this work are presented on very simple models on purpose. We think that they can be used for many variants of Equation (1), changing for example the boundary conditions, or the cross section.

There is no a-priori reason why it should not also work in nonlinear situations, provided that uniform in time smoothness estimates are known for the solution of the problem under study (such estimates are often difficult to obtain for general data, but they can sometimes be proven in special regimes).

Note that the challenging problems of cross sections σ such that $\sigma = 0$ on a set of strictly positive measure is not treated here. We refer to [2] for cases of degeneracy in the simpler situation of coercivity (in the context of reaction-diffusion equations).

The rest of the paper is structured in this way: in Section 2, the theory of existence, uniqueness and smoothness for the linear transport and Goldstein-Taylor models is briefly presented and the hypocoercivity method using differential inequalities is recalled. Section 3 is devoted to the study of the Goldstein-Taylor system. A priori estimates are first presented in Subsection 3.1, then the large time behavior is investigated in Subsection 3.2. The case in which the data are very smooth is treated in Subsection 3.3. Finally, the proof of Theorem 1.1 is presented in Section 4.

2. Preliminaries

We begin with the following proposition about the Cauchy problem for systems (1)-(2) and (3)-(4).

Proposition 2.1. *Consider the transport system* (1)-(2) *with initial data $f_0 \in L^1(\mathbb{T} \times V)$. Then there exists a unique (generalized) solution $f \in C(\mathbb{R}_+; L^1(\mathbb{T} \times V))$ to this problem.*

Consider then the Goldstein-Taylor model (3)-(4) *with initial data $(u_0, v_0) \in L^1(\mathbb{T}) \times L^1(\mathbb{T})$. Then there exists a unique (generalized) solution (u, v) to this system in $C(\mathbb{R}_+; L^1(\mathbb{T}) \times L^1(\mathbb{T}))$.*

Proof of Proposition 2.1: It is well known that the unbounded operator

$$B(u, v) = \left(-\frac{\partial u}{\partial x} + \sigma(x)(v - u), \frac{\partial v}{\partial x} + \sigma(x)(u - v) \right)$$

on $L^1(\mathbb{T}) \times L^1(\mathbb{T})$ with domain

$$\mathcal{D}(B) = \{(u, v) \in W^{1,1}(\mathbb{T}) \times W^{1,1}(\mathbb{T})\}$$

is dissipative (see, for example, [8], [10]). The existence and uniqueness of the solution (u, v) easily follows from the method presented in [5]. The same kind of arguments holds for the transport model. This concludes the proof of Proposition 2.1.

Then, we introduce another proposition, whose proof is a direct consequence of lemma 12 in [4]. This proposition replaces Gronwall's lemma in the context of hypocoercive equations.

Proposition 2.2. *Let x and y be two nonnegative C^2 functions defined on $\mathbb{R}_+$ and satisfying (for all $t > 0$)*

$$\begin{cases} -x'(t) \geq \alpha_1\, y^{1+\delta}(t), \\ y''(t) \geq \alpha_3\, x(t) - \alpha_2 y^{1-\varepsilon}(t), \end{cases} \tag{8}$$

for some constants $\delta \geq 0$, $\varepsilon \in]0,1[$ and α_1, α_2, $\alpha_3 > 0$.

Then there exists a constant $\alpha_4 > 0$ depending only on $x(0)$, α_1, α_2, α_3, δ and ε such that (for all $t > 0$),

$$x(t) \leq \alpha_4\, t^{-\frac{1-\varepsilon}{\delta+\varepsilon}}.$$

3. The Golstein-Taylor model

3.1. **A priori estimates.** We begin with a result of boundedness in L^2 for derivatives of first order in x and second order in t of the solutions of system (3)-(4).

Proposition 3.1. *Let $u_0, v_0 \in H^2(\mathbb{T})$, and $\sigma \in H^1(\mathbb{T})$. Then, there exists a constant γ (depending explicitly on $||u_0||_{H^2}$, $||v_0||_{H^2}$ and $||\sigma||_{H^1}$) such that the solution (u, v) of system (3)-(4) satisfies the bound*

$$\sup_{t\geq 0} \int_{\mathbb{T}} \left((\partial_t u)^2 + (\partial_{tt} u)^2 + (\partial_x u)^2 \right) dx \leq \gamma,$$

$$\sup_{t\geq 0} \int_{\mathbb{T}} \left((\partial_t v)^2 + (\partial_{tt} v)^2 + (\partial_x v)^2 \right) dx \leq \gamma.$$

Proof of Proposition 3.1: We differentiate k times ($k = 1, 2$) the equations of system (3) with respect to the variable t, and multiply the result by $2\partial^k u/\partial t^k$ and $2\partial^k v/\partial t^k$ respectively. After integrating with respect to $x \in \mathbb{T}$, we end up with

$$\frac{d}{dt} \int_{\mathbb{T}} \left[\left(\frac{\partial^k u}{\partial t^k} \right)^2 + \left(\frac{\partial^k v}{\partial t^k} \right)^2 \right] dx = -2 \int_{\mathbb{T}} \sigma(x) \left[\frac{\partial^k u}{\partial t^k} - \frac{\partial^k v}{\partial t^k} \right]^2 dx \leq 0. \tag{9}$$

Then, we observe that

$$\partial_{tt} u = \partial_{xx} u + 2\,\sigma\, \partial_x u + \sigma'\,(u - v) + 2\,\sigma^2\,(u - v)$$

and

$$\partial_{tt} v = \partial_{xx} v - 2\,\sigma\, \partial_x v + \sigma'\,(u - v) + 2\,\sigma^2\,(v - u).$$

We know that at time 0, $\partial_{xx} u \in L^2$, $\sigma \in H^1 \subset L^\infty$, so that $\sigma\, \partial_x u \in L^2$, $\sigma^2\,(v - u) \in L^2$, and $u \in H^2 \subset L^\infty$ so that $\sigma'(u - v) \in L^2$.

Finally, $\partial_{tt} u(0,\cdot) \in L^2$, and so do $\partial_t u(0,\cdot)$, $\partial_t v(0,\cdot)$ and $\partial_{tt} v(0,\cdot)$. Using estimate (9), we obtain

$$\sup_{t\geq 0} \int_{\mathbb{T}} \left((\partial_t u)^2 + (\partial_{tt} u)^2 + (\partial_t v)^2 + (\partial_{tt} v)^2 \right) dx < +\infty.$$

We finally observe that

$$\partial_x u = -\partial_t u + \sigma(v-u), \qquad \partial_x v = \partial_t v + \sigma(v-u).$$

Using the bound above and the fact that $\sigma \in H^1 \subset L^\infty$, we can conclude the proof of Proposition 3.1.

Next proposition generalizes to derivatives of any order the results of Proposition 3.1.

Proposition 3.2. *Let $k \in \mathbb{N}^*$, $u_0, v_0 \in H^k(\mathbb{T})$, and $\sigma \in W^{k-1,\infty}(\mathbb{T})$.*

Then there exists a constant γ_k depending explicitly on $||u_0||_{H^k(\mathbb{T})}$, $||v_0||_{H^k(\mathbb{T})}$, $||\sigma||_{W^{k-1,\infty}(\mathbb{T})}$ such that the solution (u,v) of system (3)-(4) *satisfies the bound*

$$\sup_{t\geq 0} \int_{\mathbb{T}} \left[\left(\nabla^k_{t,x} u\right)^2 + \left(\nabla^k_{t,x} v\right)^2 \right] (t,x)\, dx \leq \gamma_k.$$

Proof of Proposition 3.2: Estimate (9) still holds. In fact, using the same proof, it holds for any $k \in \mathbb{N}^*$. Observing that using the equations, the derivatives of order k in time of (u,v) at time $t=0$ are sums of terms of the form $(\prod_{l=0,..,k-1} (\partial_x^l \sigma(x))^{p_l})\, \partial_x^m u(0,x)$ (and $(\prod_{l=0,..,k-1} (\partial_x^l \sigma(x))^{p_l})\, \partial_x^m v(0,x))$ with $m \leq k$ (and $p_l \in \mathbb{N}$), we see that the derivatives $\frac{\partial^k u}{\partial t^k}$, $\frac{\partial^k v}{\partial t^k}$ lie in $L^\infty(\mathbb{R}_+; L^2(\mathbb{T}))$.

Then, differentiating the equations $k-1$ times with respect to t and one time with respect to x, we see that

$$\frac{\partial^k u}{\partial t^k} = -\frac{\partial}{\partial x} \frac{\partial^{k-1} u}{\partial t^{k-1}} + \sigma \frac{\partial^{k-1}}{\partial t^{k-1}} (v-u).$$

The left-hand side of this formula belongs to $L^\infty(\mathbb{R}_+; L^2(\mathbb{T}))$, and so does the second part of the right-hand side. As a consequence, $\frac{\partial}{\partial x}\frac{\partial^{k-1} u}{\partial t^{k-1}}$ also lies in $L^\infty(\mathbb{R}_+; L^2(\mathbb{T}))$.

Differentiating then the equation $k-2$ times with respect to t and two times with respect to x, we obtain that $\frac{\partial^2}{\partial x^2}\frac{\partial^{k-2} u}{\partial t^{k-2}}$ lies in $L^\infty(\mathbb{R}_+; L^2(\mathbb{T}))$.

A simple induction enables to conclude the proof of Proposition 3.2.

3.2. **Asymptotic behavior.** This subsection is devoted to the study of the long-time asymptotics for the solution (u,v) of (3)-(4) under the assumption that the initial data are of class $H^2(\mathbb{T})$ and that the cross section σ belongs to $H^1(\mathbb{T})$.

Proof of the first part of Theorem 1.2: We introduce the quantity

$$H(t) = \int_{\mathbb{T}} \left[(u-u_\infty)^2 + (v-v_\infty)^2\right] \, dx \tag{10}$$

which measures the distance of (u, v) to the global equilibrium (u_∞, v_∞). We also introduce

$$K(t) = \int_{\mathbb{T}} (u-v)^2 \, dx, \tag{11}$$

which measures the distance of (u, v) to the set of local equilibria (that is, the set of functions (u, v) which depend on x and such that $u = v$).

The following result holds:

Lemma 3.1. *We assume that* $(u := u(t,x), v := v(t,x))$ *is the (unique) solution to Equations* (3)-(4) *with initial data* u_0, v_0 *lying in* $H^2(\mathbb{T})$*, and for a cross section* σ *satisfying Assumption* 1 *and belonging to* $H^1(\mathbb{T})$*. Then, the quantities* H, K *defined in* (10)-(11) *satisfy the following system of differential inequalities:*

$$-\frac{d}{dt} H(t) \geq \alpha_1 \, K^{1+\frac{\lambda_\sigma}{2}}(t), \tag{12}$$

$$\frac{d^2}{dt^2} K(t) \geq 2\, H(t) - \alpha_2 \, K^{1/2}(t), \tag{13}$$

where α_1 *depends on* C_σ *and* γ*, and* α_2 *depends on* $||\sigma||_{H^1}$ *and* γ*. Both those coefficients can be estimated explicitly in terms of those quantities.*

Proof of Lemma 3.1: For the sake of simplicity, the proof is written when σ satisfies Assumption 1 with $N = 1$, and $x_1 = 0$, so that

$$\sigma(x) \geq C_\sigma \, |x|^{\lambda_\sigma}.$$

The general case can be treated without further difficulties.

Along the flow of Equation (3), the derivative of H is given by

$$\frac{d}{dt} H(t) = -2 \int_{\mathbb{T}} \sigma(x)[u(x,t) - v(x,t)]^2 \, dx. \tag{14}$$

Then, we observe that for any $h \in]0, 1/4[$,

$$\int_{\mathbb{T}} |u-v|^2(t,x)\, dx \leq \int_{|x|\leq h} |u-v|^2(t,x)\, dx + \int_{h\leq |x| \leq 1/2} |u-v|^2(t,x)\, dx$$

$$\leq 2 \int_{|x|\leq h} |u-v|^2(t, x + \frac{x}{|x|}\, h)\, dx$$

$$+2 \int_{|x|\leq h} |(u-v)(t, x + \frac{x}{|x|}\, h) - (u-v)(t,x)|^2 \, dx + \int_{h \leq |x| \leq 1/2} |u-v|^2(t,x)\, dx$$

$$\leq 2\int_{|x|\leq h}\frac{\sigma(x+\frac{x}{|x|}\,h)}{C_\sigma\,|x+\frac{x}{|x|}\,h|^{\lambda_\sigma}}\,|u-v|^2(t,x+\frac{x}{|x|}\,h)\,dx \tag{15}$$

$$+2\,h^2\int_{|x|\leq h}\left|\int_{\theta=0}^1\left(\partial_x(u-v)(t,x+\theta\,\frac{x}{|x|}\,h)\right)d\theta\right|^2 dx$$

$$+\int_{h\leq|x|\leq 1/2}\frac{\sigma(x)}{C_\sigma\,|x|^{\lambda_\sigma}}\,|u-v|^2(t,x)\,dx$$

$$\leq\frac{3}{C_\sigma\,h^{\lambda_\sigma}}\int_{\mathbb{T}}\sigma(x)\,|u-v|^2(t,x)\,dx+2\,h^2\int_{\mathbb{T}}|\partial_x(u-v)|^2(t,x)\,dx.$$

Optimizing with respect to h (and distinguishing between the cases $h\leq\frac14$ and $h>\frac14$), we end up with

$$K(t)\leq Cst(\gamma,C_\sigma)\left(\int_{\mathbb{T}}\sigma(x)\,|u-v|^2(t,x)\,dx\right)^{\frac{1}{1+\frac{\lambda_\sigma}{2}}}.$$

This, together with (14), leads immediately to estimate (12).

In order to obtain inequality (13), we need to compute the second derivative of K with respect to time along the flow of system (3):

$$\frac{d^2}{dt^2}K(t)=2\int_{\mathbb{T}}(v_t-u_t)^2\,dx+2\int_{\mathbb{T}}(v-u)(v_{tt}-u_{tt})\,dx.$$

By using system (3), we therefore deduce

$$\frac{d^2}{dt^2}K(t)=2\int_{\mathbb{T}}[v_x+u_x+2\sigma(x)(u-v)]^2\,dx$$

$$+2\int_{\mathbb{T}}(v-u)(v_{tt}-u_{tt})\,dx,$$

which implies

$$\frac{d^2}{dt^2}K(t)\geq\int_{\mathbb{T}}[(u+v)_x]^2\,dx-8\,||\sigma||_{L^\infty}\int_{\mathbb{T}}(u-v)^2\,dx \tag{16}$$

$$-\left(\int_{\mathbb{T}}(u-v)^2\,dx\right)^{1/2}\left(\int_{\mathbb{T}}(u_{tt}-v_{tt})^2\,dx\right)^{1/2}.$$

Thanks to Poincaré-Wirtinger's inequality,

$$\int_{\mathbb{T}}[(u+v)_x]^2\,dx\geq\int_{\mathbb{T}}\left(u+v-(u_\infty+v_\infty)\right)^2\,dx=2\,H(t)-\int_{\mathbb{T}}(u-v)^2\,dx. \tag{17}$$

As a consequence, and using a Sobolev inequality,

$$\frac{d^2}{dt^2}K(t)\geq 2\,H(t)-(1+8\,||\sigma||_{L^\infty})\,K(t)-(2\,\gamma^{1/2})\,K(t)^{1/2}$$

$$\geq 2\,H(t) - Cst(\gamma, ||\sigma||_{H^1})\,K(t)^{1/2}.$$

Thus we get (13), and this concludes the proof of Proposition 3.1.

Formula (6) in Theorem 1.2 is then a simple consequence of Proposition 3.1 and Proposition 2.2. This concludes the proof of the first part of Theorem 1.2.

3.3. **Extra Smoothness of the data.**

Proof of the second part of Theorem 1.2: We now treat the case when σ as well as the initial data u_0, v_0 lie in $C^\infty(\mathbb{T})$. We begin with the

Lemma 3.2. *We assume that $(u := u(t,x), v := v(t,x))$ is the (unique) solution to system* (3)-(4) *with initial data u_0, v_0 lying in $C^\infty(\mathbb{T})$, and for a cross section σ satisfying Assumption* 1 *and belonging to $C^\infty(\mathbb{T})$. Then, the quantities H, K defined in* (10)-(11) *satisfy the following system of differential inequalities (for any $\varepsilon \in]0,1[$):*

$$-\frac{d}{dt}H(t) \geq \alpha_4\,K(t)^{1+\frac{\lambda_\sigma}{3}}, \tag{18}$$

$$\frac{d^2}{dt^2}K(t) \geq H(t) - \alpha_5\,K(t)^{1-\varepsilon}, \tag{19}$$

where α_4 depends on C_σ and γ_i $(i = 0, 2)$, and α_5 depends on ε, $||\sigma||_{L^\infty}$, γ_0 and $\gamma_{1+[\frac{1}{\varepsilon}]}$. Both those coefficients can be estimated explicitly in terms of these quantities.

Proof of Lemma 3.2: We still have (14) and (15). However, we then proceed in the computation in the following way:

$$\int_{\mathbb{T}} |u-v|^2(t,x)\,dx \leq \frac{3}{C_\sigma\,h^{\lambda_\sigma}} \int_{\mathbb{T}} \sigma(x)\,|u-v|^2(t,x)\,dx + 2\,h^3\,||(u-v)_x(t,\cdot)||^2_{L^\infty(\mathbb{T})}$$

$$\leq \frac{3}{C_\sigma\,h^{\lambda_\sigma}} \int_{\mathbb{T}} \sigma(x)\,|u-v|^2(t,x)\,dx + Cst(\gamma_1,\gamma_2)\,h^3,$$

thanks to a Sobolev inequality.

After optimizing with respect to h, we end up with

$$\int_{\mathbb{T}} |u-v|^2(t,x)\,dx \leq Cst(\gamma_0,\gamma_2,C_\sigma) \left(\int_{\mathbb{T}} \sigma(x)\,|u-v|^2(t,x)\,dx \right)^{\frac{1}{1+\frac{\lambda_\sigma}{3}}},$$

so that (18) holds.

We now observe that estimates (16) and (17) still hold. We therefore still have

$$\frac{d^2}{dt^2}K(t) \geq 2\,H(t) - (1+8\,||\sigma||_{L^\infty})\,K(t) - K(t)^{1/2} \left(\int_{\mathbb{T}} (u_{tt}-v_{tt})^2\,dx \right)^{1/2}.$$

Then, we observe that

$$u_{tt} - v_{tt} = (u-v)_{xx} + 2\sigma\,(u_x + v_x) + 4\sigma^2 (u-v).$$

By interpolation with high-order derivatives, we get (for any $\varepsilon > 0$)

$$\int_{\mathbb{T}} |u_{xx} - v_{xx}|^2(t,x)\,dx \le 2\,\gamma_{1+[\frac{1}{\varepsilon}]}^{2\varepsilon}\, K(t)^{1-2\,\varepsilon}.$$

Also by interpolation,

$$\int_{\mathbb{T}} |u_x + v_x|^2(t,x)\,dx \le 2\,\gamma_{1+[\frac{1}{\varepsilon}]}^{\varepsilon} \left(\int_{\mathbb{T}} \left| u(t,x) + v(t,x) - (u_\infty + v_\infty) \right|^2 dx \right)^{1-\varepsilon}$$
$$\le 2\,\gamma_{1+[\frac{1}{\varepsilon}]}^{\varepsilon}\, H(t)^{1-\varepsilon}.$$

As a consequence,

$$\frac{d^2}{dt^2} K(t) \ge 2\,H(t) - Cst\,(||\sigma||_{L^\infty})\,K(t) - Cst\,(||\sigma||_{L^\infty}, \gamma_{1+[\frac{1}{\varepsilon}]}, \varepsilon)\,K(t)^{1-\varepsilon}$$
$$- Cst\,(||\sigma||_{L^\infty}, \gamma_{1+[\frac{1}{\varepsilon}]}, \varepsilon)\,K(t)^{1/2}\,H(t)^{1/2-\varepsilon/2}$$
$$\ge H(t) - Cst\,(||\sigma||_{L^\infty}, \gamma_0, \gamma_{1+[\frac{1}{\varepsilon}]}, \varepsilon)\,K(t)^{1-\varepsilon}.$$

This concludes the proof of Proposition 3.2.

Formula (7) in Theorem 1.2 is then a simple consequence of Proposition 3.2 and Proposition 2.2. This concludes the proof of Theorem 1.2.

4. The Linear Transport Equation

Proof of Theorem 1.1: We observe first that for $k = 0, 1, 2$,

$$\frac{d}{dt} \int_{\mathbb{T}^d} \int_V \frac{1}{2} \left| \frac{\partial^k f}{\partial t^k} \right|^2 dv dx = -\frac{1}{2} \int_{\mathbb{T}^d} \sigma(x) \int_V \left| \frac{\partial^k \bar{f}}{\partial t^k} - \frac{\partial^k f}{\partial t^k} \right|^2 dv dx.$$

As a consequence, we see that (for $k = 0, 1, 2$), $\frac{\partial^k f}{\partial t^k} \in L^\infty(\mathbb{R}_+; L^2(\mathbb{T}^d \times V))$, as soon as $\frac{\partial^k f}{\partial t^k}(0) \in L^2(\mathbb{T}^d \times V)$. Using the identity

$$\frac{\partial^2 f}{\partial t^2} = v \otimes v : \nabla_x \nabla_x f - \sigma\,(v \cdot \nabla_x \bar{f}) + 2\,\sigma\,(v \cdot \nabla_x f)$$
$$-\sigma\,\nabla_x \cdot (\overline{v\,f}) - \sigma^2\,(\bar{f} - f) + (-v \cdot \nabla_x \sigma)\,(\bar{f} - f),$$

we see (thanks to the assumptions of Theorem 1.1) that it is the case.

Then, we observe that f satisfies the maximum principle, so that $f \in L^\infty(\mathbb{R}_+ \times \mathbb{T}^d \times V)$.

We now introduce the quantities

$$H(f) = \int\int |f - f_\infty|^2\,dv dx, \qquad K(f) = \int\int |f - \bar{f}|^2\,dv dx.$$

We observe that

$$\frac{dH(f)}{dt} = -2 \int \sigma(x) \int |f - \bar{f}|^2 \, dvdx.$$

We interpolate in this way:

$$\int \int |f - \bar{f}|^2 \, dvdx \leq \int_{|x| \leq h} \int |f - \bar{f}|^2 \, dvdx + \int_{|x| \geq h} \int |f - \bar{f}|^2 \, dvdx$$

$$\leq 2\, h \, ||f||^2_{L^\infty} + \frac{C_\sigma}{|h|^{\lambda_\sigma}} \int \sigma \, |f - \bar{f}|^2 \, dvdx.$$

Optimizing in h, we get

$$-\frac{dH(f)}{dt} \geq 2 \int \sigma \, |f - \bar{f}|^2 \, dvdx \geq Cst \; K(f)^{1+\lambda_\sigma}. \tag{20}$$

We now treat the second derivative in time of $K(f)$.

We see that

$$\frac{d^2}{dt^2} K(f) = 2 \int \int \left| \frac{\partial f}{\partial t} - \frac{\partial \bar{f}}{\partial t} \right|^2 \, dvdx + 2 \int \int (f - \bar{f}) \left(\frac{\partial^2 f}{\partial t^2} - \frac{\partial^2 \bar{f}}{\partial t^2} \right) dvdx$$

$$\geq \int \int |\nabla_x (\overline{vf} - vf)|^2 \, dvdx - Cst \; \int |f - \bar{f}|^2 \, dvdx$$

$$- \; Cst \left(\int |f - \bar{f}|^2 \, dvdx \right)^{1/2} \left\| \frac{\partial^2 f}{\partial t^2} \right\|_{L^\infty(\mathbb{R}_+; L^2(\mathbb{T}^d \times V))}$$

$$\geq Cst \; \int \int \left| \overline{vf} - vf - \int_{\mathbb{T}} (\overline{vf} - vf) \, dx \right|^2 dvdx - Cst \; (K(f) + K(f)^{1/2})$$

$$\geq Cst \; \int \int |v|^2 \left| \bar{f} - \int_{\mathbb{T}} \bar{f} \, dx \right|^2 dvdx - Cst \; (K(f) + K(f)^{1/2})$$

$$\geq Cst \; H(f) - Cst \; K(f)^{1/2}.$$

We end up with the differential inequality

$$\frac{d^2}{dt^2} K(f) \geq Cst \; H(f) - Cst \; K(f)^{1/2}. \tag{21}$$

Estimates (20) and (21) enable to use Proposition 2.2, and lead to the estimate

$$H(f)(t) \leq Cst \; t^{-\frac{1}{1+2\,\lambda_\sigma}}.$$

This ends the proof of Theorem 1.1.

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