OPTIMAL ESTIMATE OF THE SPECTRAL GAP FOR THE DEGENERATE GOLDSTEIN-TAYLOR MODEL

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Abstract. In this paper we study the decay to the equilibrium state for the solution of a generalized version of the Goldstein-Taylor system, posed in the one-dimensional torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, by allowing that the nonnegative cross section σ can vanish in a subregion $X := \{x \in \mathbb{T} \,|\, \sigma(x) = 0\}$ of the domain with $\operatorname{meas}(X) \geq 0$ with respect to the Lebesgue measure.

We prove that the solution converges in time, with respect to the strong L^2-topology, to its unique equilibrium with an exponential rate whenever $\operatorname{meas}(\mathbb{T} \setminus X) \geq 0$ and we give an optimal estimate of the spectral gap.

1. Introduction

The investigation about explicit rates of approach to equilibrium in large time, for kinetic equations, is an active field of research and many results have been obtained, both in the linear and in the non linear case.

An important concept, in this context, is *hypocoercivity*. This property appears in many evolution equations which have a conservative part and a dissipative one. Even if the conservative part alone does not induce relaxation and the dissipative one is not sufficient to induce convergence to equilibrium, sometimes the combination of the two parts leads to relaxation. When this situation occurs, the equation is said to be *hypocoercive*.

For kinetic equations, the conservative term is usually the free transport operator, which mixes the space and the velocity variables, whereas the dissipative part is a collision operator, whose null space does not depend on the space variable.

Furthermore, the key ingredient of many proofs is based upon the independence of the null space of the dissipative operator on the space variable. This allows indeed a local control of the dissipative properties of the equation, and hence the solution is locally "attracted" everywhere toward its local equilibrium (see, for instance, [11, 7, 5, 16]).

The situation is, however, quite different in the degenerate case, i.e. when the collision operator can vanish in the spatial domain of the problem (even if the degeneracy happens only at isolated points). In the region of degeneracy, indeed, the null space of the collision operator becomes trivial.

This problem has been studied for the first time by Desvillettes and the second author in [4]. In this article, they proved, under very stringent hypotheses on the degeneracy of the cross section, that the solution of the linear Boltzmann equation and the solution of a reduced two-velocity model, namely the generalized Goldstein-Taylor system, converge in time to their equilibrium with (at least) polynomial speed. They also conjectured that some explicit rate should still exist even in more general degenerate situations.

The deep reason of this conjecture is based on the fact that the hypocoercivity properties, which are locally lost in the regions of degeneracy, can be recovered

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at a global level, through the action of the mixing term, here the free transport operator.

However, at least in more than one space dimension, the decay rate to equilibrium cannot be – in general – exponential, as shown by a counterexample of the authors in [2].

Indeed, we proved that the linear Boltzmann equation on the torus $\mathbb{T}^d$, $d \geq 2$, with velocities on the sphere $\mathbb{S}^{d-1}$, has, for a wide class of cross sections in L^∞, a L^2-distance to equilibrium that cannot decay faster than $t^{-1/2}$.

The exponential rate can be recovered only by assuming an additional hypothesis on the geometrical structure of the cross section.

This additional requirement has been introduced by the authors in [3] and has been called the *geometrical condition*, as a reminiscence of the Bardos-Lebeau-Rauch condition that guarantees the exponential stabilization of the telegrapher's equation [1]. We proved that the geometrical condition is necessary and sufficient to recover the exponential decay in time to equilibrium. However, our proof in [3] is not constructive because it is based on a compactness argument.

Hence, the problem of finding an explicit exponential rate for the linear Boltzmann equation, with cross sections satisfying the geometrical condition, is still an open problem. Likewise, finding the best convergence rate to equilibrium in the general case is still open.

The aim of this article is to go beyond the actual state-of-the-art and give a quantitative study of the spectral gap in the degenerate case. Here we will restrict ourselves to consider the simplest possible degenerate kinetic equation.

We prove that the convergence to equilibrium for the generalized Goldstein-Taylor system proposed in [4] is exponential in time and, moreover, we can exactly characterize the convergence rate. The geometrical condition of [3] is automatically fulfilled here, since the spatial domain is one-dimensional. Consequently, our result agrees with the general theory.

The proof is based on the equivalent formulation of the problem in terms of the telegrapher's equation, by following a trick by Kac [8], and on a careful estimate of the time decay of the energy for the non-homogeneous telegrapher's equation, proved by Rauch and Taylor in [13, 14] (see also [9]).

In this paper we completely characterize the spectral gap of the degenerate Goldstein-Taylor system, with general degenerate cross section, in terms of the geometrical properties of the cross section itself.

Because of the peculiar situation of the one-dimensional Euclidean space, all uniformly bounded cross sections that differ from the null function (in L^1 sense) satisfy the geometrical condition, and hence they generate hypocoercive effects in agreement with our general result in [3].

Our result is optimal and provides the definitive answer to the estimation of the spectral gap of the Goldstein-Taylor model.

Part of the proof heavily depends on the fact that the set of admissible velocities is discrete.

In the case of a continuous set of velocities, a similar result would need new techniques of proof because the set of the moment equations is no longer closed.

The paper is organized as follows: in Section 2 we precisely state our problem and then, in Section 3, we prove our main theorem about the long-time behaviour of the problem.

2. The problem and its basic properties

We consider a simplified one-dimensional version of the linear Boltzmann equation in which only two velocities are allowed.

It is a variant of the well-known Goldstein-Taylor model [6, 15]. It describes the behaviour of a gas composed of two kinds of particles moving parallel to the x-axis with constant speeds, of equal modulus $c = 1$, the first one in the positive x-direction with density $u := u(x, t)$, the other one in the negative x-direction with density $v := v(x, t)$. Both types of particles experience switches in velocity, distributed under a Poisson law. The corresponding system of equations is:

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial x} = \sigma(x)(v - u) \\ \\ \dfrac{\partial v}{\partial t} - \dfrac{\partial v}{\partial x} = \sigma(x)(u - v), \end{cases} \tag{2.1}$$

where $x \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$ and $t \geq 0$.

Such set of equations will satisfy the initial conditions

$$u(0, x) = u^{\mathrm{in}}(x), \qquad v(0, x) = v^{\mathrm{in}}(x). \tag{2.2}$$

We will suppose henceforth that $(u^{\mathrm{in}}, v^{\mathrm{in}}) \in H^1(\mathbb{T}) \times H^1(\mathbb{T})$.

The cross section σ describes possible anisotropy effects. In what follows, we will suppose that

$$\sigma \in L^\infty(\mathbb{T}), \text{ with } \sigma \geq 0 \text{ a.e. and } \int_{\mathbb{T}} \sigma(x)\,\mathrm{d}x > 0. \tag{2.3}$$

We now state our main result:

Theorem 2.1. *Let $(u^{\mathrm{in}}, v^{\mathrm{in}}) \in H^1(\mathbb{T}) \times H^1(\mathbb{T})$ be nonnegative functions and let $\sigma \in L^\infty(\mathbb{T})$ satisfy* (2.3)*. Denote also*

$$u_\infty := \frac{1}{2}\int_{\mathbb{T}} (u^{\mathrm{in}} + v^{\mathrm{in}})\,\mathrm{d}x.$$

Then, there exists a positive constant A_*, depending on $||u^{\mathrm{in}}||_{H^1(\mathbb{T})}$, $||v^{\mathrm{in}}||_{H^1(\mathbb{T})}$ and $||\sigma||_{L^\infty(\mathbb{T})}$, such that the solution (u, v) of the Goldstein-Taylor model (2.1)-(2.2) *satisfies the inequality*

$$H(t) := \|u - u_\infty\|^2_{L^2(\mathbb{T})} + \|v - u_\infty\|^2_{L^2(\mathbb{T})} \leq A_* \exp(-\alpha t), \tag{2.4}$$

where

$$\alpha = 2\|\sigma\|_{L^1(\mathbb{T})}.$$

Moreover, the decay rate α is optimal in the following sense:

$$\alpha = \sup\left\{\beta \geq 0 \,:\, \forall t \geq 0,\ \forall (u^{\mathrm{in}}, v^{\mathrm{in}}) \in H^1(\mathbb{T}) \times H^1(\mathbb{T}),\ H(t) \leq A_* e^{-\beta t}\right\}.$$

3. Proof of the main result

We now prove the main result of this article, namely the exponential decay in time of the solution (u, v) of the generalized Goldstein-Taylor system (2.1)-(2.2) to the stationary state (u_∞, u_∞).

The proof is based on the fact that the quantity

$$H(t) := \int_{\mathbb{T}} [(u - u_\infty)^2 + (v - u_\infty)^2]\,\mathrm{d}x,$$

already defined in Theorem 2.1, is a Lyapunov functional of the system and is controlled by the energy of the telegrapher's equation (defined in (3.7) below). A quantitative estimate on the long-time decay of H will give directly the asymptotic behaviour of the L^2 distance between the solution of (2.1)-(2.2) and the stationary state.

Finally, the optimality of our result will be tested on a particular solution of the generalized Goldstein-Taylor equations (2.1)-(2.2).

We underline that the results presented below need different regularity requirements.

For this reason, we will write explicitly the hypotheses on the regularity of the initial conditions and of the cross section in the statement of the corresponding theorems.

3.1. **A priori estimates.** In order to make the paper self-consistent, we summarize here some basic properties of the generalized Goldstein-Taylor system (2.1)-(2.2). The first result concerns the well-posedness of the problem.

Proposition 3.1. *Consider the Goldstein-Taylor model* (2.1)-(2.2) *with nonnegative initial data* $(u^{\mathrm{in}}, v^{\mathrm{in}}) \in L^1(\mathbb{T}) \times L^1(\mathbb{T})$ *and with cross section* $\sigma \in L^\infty(\mathbb{T})$ *such that* $\sigma \geq 0$ *a.e.. Then there exists a unique (generalized) nonnegative solution* (u, v) *of this system in* $C(\mathbb{R}^+; L^1(\mathbb{T}))^2$.

Proof. The result is immediate since the collision term can be treated as a bounded perturbation of the transport operator [12]. □

We then give some a-priori estimates of the Cauchy problem (2.1)-(2.2). The following lemma holds:

Lemma 3.2. *Let* (u, v) *be the solution of the generalized Goldstein-Taylor system* (2.1)-(2.2)*, with nonnegative initial data* $(u^{\mathrm{in}}, v^{\mathrm{in}}) \in L^\infty(\mathbb{T}) \times L^\infty(\mathbb{T})$ *and with nonnegative cross section* $\sigma \in L^\infty(\mathbb{T})$*. Then, for any smooth convex function* $\varphi(r)$*,* $r \geq 0$*, we have*

$$\int_{\mathbb{T}} [\varphi(u(t,x)) + \varphi(v(t,x))] \, \mathrm{d}x \leq \int_{\mathbb{T}} \left[\varphi(u^{\mathrm{in}}(x)) + \varphi(v^{\mathrm{in}}(x))\right] \, \mathrm{d}x \tag{3.1}$$

for all $t > 0$.

In particular, the conservation of mass and the maximum principle hold, i.e.

$$\|u(t,\cdot) + v(t,\cdot)\|_{L^1(\mathbb{T})} = \|u^{\mathrm{in}} + v^{\mathrm{in}}\|_{L^1(\mathbb{T})}$$

and

$$\max\{\|u(t,\cdot)\|_{L^\infty(\mathbb{T})}, \|v(t,\cdot)\|_{L^\infty(\mathbb{T})}\} \leq \|u^{\mathrm{in}} + v^{\mathrm{in}}\|_{L^\infty(\mathbb{T})}.$$

Proof. We apply the strategy of [10]. Let $\varphi(r)$, $r \geq 0$, a smooth convex function. We multiply the first equation of (2.1) by $\varphi'(u)$ and the second one by $\varphi'(v)$. We integrate on $\mathbb{T}$ and, by summing the two equations, we obtain

$$\frac{d}{dt} \int_{\mathbb{T}} [\varphi(u) + \varphi(v)] \, \mathrm{d}x = -\int_{\mathbb{T}} \sigma(x)(u - v) \left[\varphi'(u) - \varphi'(v)\right] \mathrm{d}x.$$

Since $\varphi \in C^2(\Omega)$ is a convex function, then φ' is monotone. Hence

$$\sigma(x)(u - v)[\varphi'(u) - \varphi'(v)] \geq 0$$

because σ is nonnegative and then, for all $t > 0$,

$$\int_{\mathbb{T}} [\varphi(u(t,x)) + \varphi(v(t,x))] \, \mathrm{d}x \leq \int_{\mathbb{T}} \left[\varphi(u^{\mathrm{in}}(x)) + \varphi(v^{\mathrm{in}}(x))\right] \mathrm{d}x.$$

By choosing $\varphi(r) = r$, we obtain the conservation of mass:

$$\int_{\mathbb{T}} [u(t,x) + v(t,x)] \, \mathrm{d}x = \int_{\mathbb{T}} \left[u^{\mathrm{in}}(x) + v^{\mathrm{in}}(x)\right] \mathrm{d}x. \tag{3.2}$$

By taking $\varphi(r) = r^p$ for all $p \geq 1$, we obtain the boundedness of any L^p-norm

$$\int_{\mathbb{T}} [u(t,x)^p + v(t,x)^p] \, \mathrm{d}x \leq \int_{\mathbb{T}} \left[u^{\mathrm{in}}(x)^p + v^{\mathrm{in}}(x)^p\right] \mathrm{d}x.$$

Moreover, when $p \to +\infty$, the lemma implies also the L^∞ bound:

$$0 \leq u(t,x), v(t,x) \leq \|u^{\mathrm{in}} + v^{\mathrm{in}}\|_{L^\infty(\mathbb{T})}. \tag{3.3}$$

□

We give here a result, whose proof is similar to Proposition 3.1 of [4], showing that the H^1-regularity of the solution of the problem (2.1)-(2.2) is preserved in time.

This result will be needed in the proof of our main theorem, which is based on the use of Poincaré-type inequalities.

Proposition 3.3. *Let $(u^{\rm in}, v^{\rm in}) \in H^1(\mathbb{T}) \times H^1(\mathbb{T})$. Then, there exists a constant γ (depending explicitly on $||u^{\rm in}||_{H^1}$, $||v^{\rm in}||_{H^1}$ and $||\sigma||_{L^\infty}$) such that the solution (u, v) of system* (2.1)-(2.2) *satisfies the bound*

$$\sup_{t\geq 0} \int_{\mathbb{T}} \Big((\partial_t u)^2 + (\partial_x u)^2 \Big)\, \mathrm{d}x \leq \gamma,$$

$$\sup_{t\geq 0} \int_{\mathbb{T}} \Big((\partial_t v)^2 + (\partial_x v)^2 \Big)\, \mathrm{d}x \leq \gamma.$$

Proof. We differentiate the equations of system (2.1) with respect to the variable t, and multiply the result by $2\partial\, u/\partial t$ and $2\partial\, v/\partial t$ respectively. After integrating with respect to $x \in \mathbb{T}$, we end up with

$$\frac{d}{dt}\int_{\mathbb{T}} \left[\left(\frac{\partial\, u}{\partial t}\right)^2 + \left(\frac{\partial\, v}{\partial t}\right)^2 \right] dx = -2 \int_{\mathbb{T}} \sigma(x) \left[\frac{\partial\, u}{\partial t} - \frac{\partial\, v}{\partial t} \right]^2 \mathrm{d}x \leq 0. \tag{3.4}$$

We finally observe that

$$\partial_x u = -\partial_t u + \sigma(v-u), \qquad \partial_x v = \partial_t v + \sigma(v-u).$$

Hence, using the bound above and the fact that $\sigma \in L^\infty$, we can conclude the proof of Proposition 3.3. □

We define the *macroscopic density*

$$\rho(t, x) := u(t, x) + v(t, v), \qquad t > 0, x \in \mathbb{T},$$

and the *flux*

$$j(t, x) := u(t, x) - v(t, v), \qquad t > 0, x \in \mathbb{T}.$$

From (2.1), it is easy to show that (ρ, j) verifies

$$\begin{cases} \partial_t \rho + \partial_x j = 0, \\ \\ \partial_t j + \partial_x \rho = -2\sigma j, \end{cases} \qquad (t, x) \in \mathbb{R}^+ \times \mathbb{T} \tag{3.5}$$

with initial conditions $\rho(0, x) = u^{\rm in}(x) + v^{\rm in}(x)$ and $j(0, x) = u^{\rm in}(x) - v^{\rm in}(x)$.

By using a trick that comes back to Kac, it can be shown that j is the solution of the telegrapher's equation.

We have indeed the following result [8].

Proposition 3.4. *Let $j = u - v$ be the flux corresponding to the Goldstein-Taylor system* (2.1)-(2.2)*. Then j is the solution of the telegrapher's equation*

$$\begin{cases} \partial^2_{tt} j - \partial^2_{xx} j + 2\sigma \partial_t j = 0, \\ \\ j(0, x) = u^{\rm in}(x) - v^{\rm in}(x), & (t, x) \in \mathbb{R}^+ \times \mathbb{T}. \\ \\ \partial_t j(0, x) = 2\sigma(x)[v^{\rm in}(x) - u^{\rm in}(x)] - \partial_x u^{\rm in}(x) - \partial_x v^{\rm in}(x), \end{cases} \tag{3.6}$$

In [14], the exponential decay in time to zero of the energy

$$E(t) := \|\partial_t j(t,\,\cdot\,)\|_{L^2(\mathbb{T})} + \|\partial_x j(t,\,\cdot\,)\|_{L^2(\mathbb{T})} \tag{3.7}$$

of the telegrapher's equation (3.6) has been proved, and explicit estimates have been provided on the decay rate [14, 13]. More precisely, the following result holds (see [9], Theorem 2):

Proposition 3.5. *Let j be the solution of the telegrapher's equation* (3.6)*, posed in the periodic torus $\mathbb{T}$.*

Then, there exists a positive constant C_*, depending on the initial data and on σ, such that

$$E(t) = \int_{\mathbb{T}} [(j_t)^2 + (j_x)^2]\,\mathrm{d}x \le C_* e^{-\alpha t},$$

where

$$\alpha = 2\|\sigma\|_{L^1(\mathbb{T})}.$$

Moreover, the decay rate α is optimal in the following sense:

$$\alpha = \sup\left\{\beta \ge 0 \,:\, \forall t \ge 0,\ \forall (u^{\mathrm{in}}, v^{\mathrm{in}}) \in H^1(\mathbb{T}) \times H^1(\mathbb{T}),\ E(t) \le C_* e^{-\beta t}\right\}.$$

3.2. **A Lyapunov functional.** The proof concerning the relationships between H and the energy of the telegrapher's equation E will be performed in several steps. The first one concerns a bound on the weighted L^2-norm of the flux.

Lemma 3.6. *There exists a constant $B \ge 0$ such that*

$$\left\|j\sqrt{\sigma}\right\|^2_{L^2(\mathbb{T})} \le BE(t),$$

where

$$B = 2\|u^{\mathrm{in}} + v^{\mathrm{in}}\|_{L^2(\mathbb{T})}.$$

Proof. We recall that

$$\partial_t j + \partial_x \rho = -2\sigma j.$$

Multiplying both sides of the equality above by j and integrating in $x \in \mathbb{T}$, we obtain immediately that

$$\int_{\mathbb{T}} j\partial_t j\,\mathrm{d}x - \int_{\mathbb{T}} \rho\partial_x j\,\mathrm{d}x = -2\int_{\mathbb{T}} \sigma j^2\,\mathrm{d}x.$$

The previous equation implies that

$$\int_{\mathbb{T}} 2\sigma j^2\,\mathrm{d}x \le \left|\int_{\mathbb{T}} j\partial_t j\,\mathrm{d}x\right| + \left|\int_{\mathbb{T}} \rho\partial_x j\,\mathrm{d}x\right|.$$

By Cauchy-Schwarz's inequality, the inequality above leads to

$$2\int_{\mathbb{T}} \sigma j^2\,\mathrm{d}x \le \|\partial_t j\|_{L^2(\mathbb{T})}\,\|j\|_{L^2(\mathbb{T})} + \|\rho\|_{L^2(\mathbb{T})}\,\|\partial_x j\|_{L^2(\mathbb{T})}$$

or

$$2\int_{\mathbb{T}} \sigma j^2\,\mathrm{d}x \le \left(\|j\|_{L^2(\mathbb{T})} + \|\rho\|_{L^2(\mathbb{T})}\right) E(t).$$

By Lemma 3.2,

$$\int_{\mathbb{T}} \sigma j^2\,\mathrm{d}x \le BE(t)$$

with

$$B := 2\left\|u^{\mathrm{in}} + v^{\mathrm{in}}\right\|_{L^2(\mathbb{T})}. \tag{3.8}$$

This ends the proof of the lemma. □

The next result gives a bound of the full L^2-norm of the flux j, which will be evaluated in terms of the energy E.

Lemma 3.7. *There exists a constant $C > 0$ such that*

$$\|j\|^2_{L^2(\mathbb{T})} \leq CE(t).$$

Moreover we have

$$C = \frac{2}{\|\sqrt{\sigma}\|^2_{L^1(\mathbb{T})}} \left[B + \frac{2}{\pi^2} \left(\|\sqrt{\sigma}\|^2_{L^1(\mathbb{T})} + \|\sigma\|_{L^1(\mathbb{T})} \right) \right].$$

Proof. First, we introduce the notation

$$\beta := \int_{\mathbb{T}} \sqrt{\sigma}(x) dx.$$

Notice that $\sqrt{\sigma}/\beta$ is a unit measure on $\mathbb{T}$. Keeping that in mind, we have

$$\begin{aligned} \|j\|^2_{L^2(\mathbb{T})} &= \int_{\mathbb{T}} (j)^2 \, \mathrm{d}x \\ &= \int_{\mathbb{T}} \left(j - \frac{1}{\beta} \int_{\mathbb{T}} j\sqrt{\sigma} \, \mathrm{d}y + \frac{1}{\beta} \int_{\mathbb{T}} j\sqrt{\sigma} \, \mathrm{d}y \right)^2 \mathrm{d}x \\ &\leq 2 \int_{\mathbb{T}} \left(j - \frac{1}{\beta} \int_{\mathbb{T}} j\sqrt{\sigma} \, \mathrm{d}y \right)^2 \mathrm{d}x + \frac{2}{\beta^2} \left(\int_{\mathbb{T}} j\sqrt{\sigma} \, \mathrm{d}x \right)^2. \end{aligned} \tag{3.9}$$

By Jensen's inequality and Lemma 3.6, we have

$$\begin{aligned} \frac{2}{\beta^2} \left(\int_{\mathbb{T}} j\sqrt{\sigma} \, \mathrm{d}x \right)^2 &\leq \frac{2}{\beta^2} \int_{\mathbb{T}} j^2 \sigma \, \mathrm{d}x \\ &= \frac{2}{\beta^2} \left\| j\sqrt{\sigma} \right\|^2_{L^2(\mathbb{T})} \\ &\leq \frac{2B}{\beta^2} E(t) \end{aligned} \tag{3.10}$$

where B is defined in (3.8).

Since $\sqrt{\sigma}/\beta$ is a unit measure on $\mathbb{T}$, the first term in the right-hand of the inequality gives

$$\begin{aligned} \left(j - \frac{1}{\beta} \int_{\mathbb{T}} j\sqrt{\sigma} \, \mathrm{d}x \right)^2 &= \left(j - \int_{\mathbb{T}} j(x) \, \mathrm{d}x - \frac{1}{\beta} \int_{\mathbb{T}} \sqrt{\sigma(x)} \left(j - \int_{\mathbb{T}} j(y) \, \mathrm{d}y \right) \mathrm{d}x \right)^2 \\ &\leq 2 \left(j - \int_{\mathbb{T}} j(x) \, \mathrm{d}x \right)^2 + \frac{2}{\beta^2} \left(\int_{\mathbb{T}} \sqrt{\sigma(x)} \left(j - \int_{\mathbb{T}} j(y) \, \mathrm{d}y \right) \mathrm{d}x \right)^2. \end{aligned}$$

Thus, by Cauchy-Schwarz's inequality and Poincaré-Wirtinger inequality,

$$\int_{\mathbb{T}} \left(j - \frac{1}{\beta} \int_{\mathbb{T}} j\sqrt{\sigma} \, \mathrm{d}x \right)^2 \mathrm{d}x \leq \frac{2}{\pi^2} \left(1 + \frac{1}{\beta^2} \|\sigma\|_{L^1(\mathbb{T})} \right) E(t). \tag{3.11}$$

Hence we obtain the result by applying inequalities (3.11) and (3.10) in inequality (3.9). □

Our last preliminary result is the control of the L^2-norm of the spatial derivative of the macroscopic density ρ in terms of the energy E.

Lemma 3.8. *There exists a constant $K > 0$ such that*

$$\|\partial_x \rho\|^2_{L^2(\mathbb{T})} \leq KE(t).$$

with

$$K := 2 + 8 \|\sigma\|^2_{L^\infty(\mathbb{T})} C,$$

where C is defined in Lemma 3.7.

Proof. We recall that

$$\partial_t j + \partial_x \rho = -2\sigma j.$$

Therefore we have:

$$|\partial_x \rho|^2 \le 2\,|\partial_t j|^2 + 8\,\|\sigma\|^2_{L^\infty(\mathbb{T})}\,|j|^2 .$$

Integrating both sides of the inequality above, we obtain

$$\|\partial_x \rho\|^2_{L^2(\mathbb{T})} \le 2\,\|\partial_t j\|^2_{L^2(\mathbb{T})} + 8\,\|\sigma\|^2_{L^\infty(\mathbb{T})}\,\|j\|^2_{L^2(\mathbb{T})}$$

thus, by Lemma 3.7, there exists $K > 0$ such that

$$\|\partial_x \rho\|^2_{L^2(\mathbb{T})} \le KE(t).$$

□

3.3. **Conclusion.** We are now ready to prove Theorem 2.1.

Proof. Assume that $u^{\mathrm{in}}, v^{\mathrm{in}} \in H^1(\mathbb{T})$. Since u_∞ is constant and nonnegative, from the definition of H given in (2.4), we obtain an equivalent formulation in terms of the macroscopic quantities ρ and j:

$$H(t) = \frac{1}{2}\left(\|\rho - 2u_\infty\|^2_{L^2} + \|j\|^2_{L^2}\right). \tag{3.12}$$

By the conservation of the mass, given in Lemma 3.2, for each $t > 0$ we have

$$\int_{\mathbb{T}} \rho(t,x)\,\mathrm{d}x = 2u_\infty.$$

Poincaré-Wirtinger's inequality hence assures that

$$\|\rho - 2u_\infty\|^2_{L^2} \le \|\partial_x \rho\|^2_{L^2} .$$

Using the inequality above in equality (3.12) implies

$$H(t) \le \frac{1}{2}\left(\|\partial_x \rho\|^2_{L^2} + \|j\|^2_{L^2}\right).$$

Consequently by Lemma 3.7 and Lemma 3.8 there exists a constant $A > 0$ such that

$$H(t) \le AE(t).$$

Thee constant A can be explicitly computed. Indeed

$$A := \frac{1}{2}(C + K),$$

where C and K are defined in Lemma 3.7 and Lemma 3.8. Proposition 3.5 allows to conclude that the inequality in Theorem 2.1 holds for every solution with u^{in}, $v^{\mathrm{in}} \in H^1(\mathbb{T})$, with $A_* = AC_*$.

In order to prove the optimality of the decay rate we consider an exact solution of the Goldstein-Taylor model (2.1)-(2.2), with initial data and cross section satisfying the hypotheses of Theorem 2.1. Then we check that its decay rate to equilibrium is exactly the rate α prescribed by the same theorem.

The procedure used for obtaining an exact solution of (2.1)-(2.2) is based on an explicit solution of the telegrapher's equation. We impose, via Theorem 3.6, that this particular solution of the telegrapher's equation corresponds to the flux of the Goldstein-Taylor model (2.1)-(2.2) for a well chosen initial datum and a suitable cross section.

It is easy to prove, by direct inspection, that the function

$$j(t,x) = e^{-k_* t}\sin(2\pi x),\ k_* > 0,\ x \in \mathbb{T} \text{ and } t \ge 0$$

solves the telegrapher's equation

$$\partial^2_{tt} j - \partial^2_{xx} j + 2k_* \partial_t j = 0$$

in the torus $\mathbb{T}$, with initial conditions

$$j(0,x) = \sin(2\pi x) \qquad \partial_t j(0,x) = -k_* \sin(2\pi x),$$

if and only if $k_* = 2\pi$.

Then, we consider (2.1)-(2.2), with constant cross section $\sigma = 2\pi$, and impose that its flux satisfies the telegrapher's equation written above, i.e. we suppose that u and v verify

$$j(t,x) = u(t,x) - v(t,x) = e^{-2\pi t}\sin(2\pi x).$$

This means that we impose that u and v solve the system of uncoupled first order partial differential equations

$$\left\{\begin{aligned} &\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = \sigma(x)(v-u) = -2\pi e^{-2\pi t}\sin(2\pi x) \\ &\frac{\partial v}{\partial t} - \frac{\partial v}{\partial x} = \sigma(x)(u-v) = 2\pi e^{-2\pi t}\sin(2\pi x), \end{aligned}\right. \tag{3.13}$$

$x \in \mathbb{T}$ and $t \geq 0$.

We now choose the particular initial conditions

$$\left\{\begin{aligned} &u^{\text{in}}(x) = \frac{1}{2}\big[\sin(2\pi x) + \cos(2\pi x)\big] + \eta \\ &v^{\text{in}}(x) = \frac{1}{2}\big[\cos(2\pi x) - \sin(2\pi x)\big] + \eta, \end{aligned}\right. \tag{3.14}$$

with $\eta \geq 2$ in order to have u^{in} and v^{in} nonnegative almost everywhere.

By direct integration of (3.13), with initial data (3.14), we hence obtain that

$$u(t,x) = \frac{e^{-2\pi t}}{2}\Bigg[\sin(2\pi(x-t))(\cos(2\pi t) - \sin(2\pi t)) + \cos(2\pi(x-t))(\cos(2\pi t) + \sin(2\pi t))\Bigg] + \eta$$

and

$$v(t,x) = \frac{e^{-2\pi t}}{2}\Bigg[\sin(2\pi(x+t))(\sin(2\pi t) - \cos(2\pi t)) + \cos(2\pi(x+t))(\cos(2\pi t) + \sin(2\pi t))\Bigg] + \eta.$$

By direct inspection and by Theorem 3.1, we can verify that u and v are the unique solution of the problem (2.1)-(2.2), with initial conditions (3.14) and constant cross section $\sigma = 2\pi$, for all $x \in \mathbb{T}$. Moreover, the functions u and v are nonnegative because the Goldstein-Taylor model preserves the sign of the initial data. The flux $j = u - v$ solves the telegrapher's equation (3.4) with initial conditions

$$j(0,x) = \sin(2\pi x) \qquad \partial_t j(0,x) = -2\pi \sin(2\pi x).$$

We can now compute an accurate lower bound of H for the particular problem (3.13)-(3.14): we notice that

$$\int_{\mathbb{T}} j^2 \,\mathrm{d}x = e^{-4\pi t}\int_{\mathbb{T}} \sin^2(2\pi x)\,\mathrm{d}x = \frac{1}{2}\exp\big(-2\|\sigma\|_{L^1(\mathbb{T})} t\big)$$

and, consequently, the estimate

$$\int_{\mathbb{T}} j^2 \,\mathrm{d}x = \int_{\mathbb{T}} (u-v)^2\,\mathrm{d}x \leq 2H(t)$$

for all $t \geq 0$ allows us to prove the optimality of the decay rate and, hence, the complete characterization of the spectral gap. □

Remark 3.9. *We do not exclude that, for some well chosen initial conditions and cross sections, the decay rate can be better than the rate indicated in Theorem 2.1. This fact does not contradict Theorem 2.1, whose result is uniform for any choices of the initial data.*

For example, if we consider the two functions

$$u = 1 + e^{-kt} \qquad v = 1 - e^{-kt}, \qquad k \in \mathbb{R}^+,\ t \geq 0,$$

it is easy to prove, by direct inspection, that they solves problem (2.1)-(2.2)*, with initial conditions*

$$u^{\rm in} = 2 \qquad v^{\rm in} = 0, \qquad \textit{for all } x \in \mathbb{T}$$

and constant cross section $\sigma = k/2$ *for all* $x \in \mathbb{T}$*.*

The asymptotic equilibrium solution is given by

$$u_\infty = \frac{1}{2} \int_{\mathbb{T}} (u^{\rm in} + v^{\rm in})\, \mathrm{d}x = 1$$

and hence

$$H(t) = \|u - u_\infty\|^2_{L^2(\mathbb{T})} + \|v - u_\infty\|^2_{L^2(\mathbb{T})} = 2e^{-2kt} = 2\exp\left(-4\|\sigma\|_{L^1(\mathbb{T})} t\right).$$

On the other hand,

$$E(t) = \|\partial_t j(t,\cdot)\|_{L^2(\mathbb{T})} + \|\partial_x j(t,\cdot)\|_{L^2(\mathbb{T})} = 4k^2 e^{-2kt} = 4k^2 \exp\left(-4\|\sigma\|_{L^1(\mathbb{T})} t\right).$$

The decay rate obtained here is hence better than the forecasts of Theorem 2.1 and Proposition 3.5. However, this behaviour is a consequence of the spatially homogeneous character of the solution. A simple Fourier analysis of the solution of the telegrapher's equation shows, indeed, that the spatial derivative in the equation is responsible of a degradation of the convergence speed to equilibrium with respect to the spatially homogeneous case.

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