H-theorem for some extensions of the Boltzmann operator

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Boltzmann operator for the four waves equation of weak turbulence theory (Zakharov)

$$Q_W(f)(v) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} W(v, v_*, v', v'_*) \Big[f(v')\, f(v'_*)\, (f(v) + f(v_*))$$

$$- f(v)\, f(v_*)\, (f(v') + f(v'_*)) \Big]$$

$$\times\, \delta_{\{v+v_* = v'+v'_*\}}\, \delta_{\{\omega(v)+\omega(v_*) = \omega(v')+\omega(v'_*)\}}\, dv_*\, dv'_*\, dv'.$$

Typical ω:

$$\omega(v) = C\, |v|^{\alpha},$$

for $0 < \alpha < 1$ and $C > 0$.

In particular, in the two-dimensional case, $\omega(v) = C \sqrt{|v|}$ is used to describe gravitational waves on a fluid surface

First part of the H-theorem for the 4-waves operator

Entropy:

$$H(f) := (-) \int_{\mathbb{R}^d} \ln f(v)\, dv;$$

Entropy production:

$$\int_{\mathbb{R}^d} Q_W(f)(v)\, f^{-1}(v)\, dv = \frac{1}{4} \int W(v, v_*, v', v'_*)$$

$$\times \Big[f^{-1}(v) + f^{-1}(v_*) - f^{-1}(v') - f^{-1}(v'_*) \Big]^2$$

$$\times\, f(v) f(v_*) f(v') f(v'_*) \delta_{\{v+v_* = v'+v'_*\}}\, \delta_{\{\omega(v)+\omega(v_*) = \omega(v')+\omega(v'_*)\}}\, dv dv_* dv'_* dv'.$$

Definition of equilibria

Definition: the equilibria of the 4-waves equation are the functions $f > 0$ such that when

$$v + v_* = v' + v'_*$$

and

$$\omega(v) + \omega(v_*) = \omega(v') + \omega(v'_*),$$

one has

$$f^{-1}(v') + f^{-1}(v'_*) = f^{-1}(v) + f^{-1}(v_*),$$

or equivalently, for $g = f^{-1}$,

$$g(v') + g(v'_*) = g(v) + g(v_*).$$

Second part of H-theorem: explicit form of equilibria

It is clear that for all $a, c \in \mathbb{R}$, $b \in \mathbb{R}^d$, the function

$$g(v) := a + b \cdot v + c\omega(v)$$

is an equilibrium.

Expected result (Second part of H-theorem): All equilibria (in a suitable functional space) have this form [except maybe for a small class of functions ω].

Natural functional space for g: weighted L^2.

Existing results

Case $\omega(v) = |v|^2$ (Boltzmann equation for monoatomic gases) :

- Proof when g is C^2 (Boltzmann);
- Proof when g is measurable, or a distribution (Truesdell-Muncaster; Wennberg)

Case $\omega(v) = \sqrt{1 + |v|^2}$ (Boltzmann equation for relativistic monoatomic gases) :

- Proof when g is C^2 (Cercignani, Kremer);
- Proof when g is a distribution (suggested in Cercignani, Kremer)

Result in the general case

Theorem (Breden, LD): Let $d \in \{2,3\}$ and $\omega \in C^2(\mathbb{R}^d - \{0\})$. Assume that there exist $i, j \in \{1, \ldots, d\}$, $i \neq j$, such that

$$\{1, \partial_i \omega, \partial_j \omega\} \text{ are linearly independant in } C^1(\mathbb{R}^d - \{0\}).$$

Assume also that the boundary ∂A of

$$A := \left\{ (v, v_*) \in \left(\mathbb{R}^d\right)^2,\ \nabla\omega(v) \neq \nabla\omega(v_*) \right\}.$$

is of measure 0 in $\left(\mathbb{R}^d\right)^2$.

Let $g \in L^1_{loc}(\mathbb{R}^d)$ be an equilibrium.

Then, there exist $a, c \in \mathbb{R}$ and $b \in \mathbb{R}^d$ such that, for a.e. v in $\mathbb{R}^d$,

$$g(v) = a + b \cdot v + c\,\omega(v).$$

Method of proof

We consider only grazing collisions, that is, collisions for which

$$v' \sim v, \qquad v'_* \sim v_*.$$

Then, the equilibria satisfy the following property (for a.e. $v, v_* \in \mathbb{R}^d$):

$$(\nabla g(v) - \nabla g(v_*)) \times (\nabla \omega(v) - \nabla \omega(v_*)) = 0.$$

This amounts to say that the entropy dissipation of the grazing collision approximation (Landau-type operator) of Q_W is zero.

The method of proof is then based on ideas taken from the study of Cercignani's conjecture for Landau's equation with Coulomb potential, cf. LD 2015; LD; Carrapatoso, LD, He 2017, using multipliers in the v_* variable.

Remark

The assumption: There exist $i, j \in \{1, \dots, d\}$, $i \neq j$, such that

$$\{1, \partial_i \omega, \partial_j \omega\} \text{ are linearly independant in } C^1(\mathbb{R}^d - \{0\})$$

is close to optimal: when it is not satified for $d = 2$, counter-examples exist.

The other assumption is probably technical.

Boltzmann operator for the three waves equation of weak turbulence theory (Zakharov)

$$Q_W(f)(v) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Big[R(v, v', v'') - R(v', v, v'') - R(v'', v, v') \Big] \, dv' dv''$$

with

$$R(v, v', v'') = W(v, v', v'')\, f(v)\, f(v')\, f(v'')\, [f^{-1}(v) - f^{-1}(v') - f^{-1}(v'')]$$

$$\times\, \delta_{\{v = v' + v''\}}\, \delta_{\{\omega(v) = \omega(v') + \omega(v'')\}}$$

and W satisfying some symmetry assumptions.

First part of the H-theorem for the 3-waves operator

Entropy:

$$H(f) := (-) \int_{\mathbb{R}^d} \ln f(v)\, dv;$$

Entropy production:

$$\int_{\mathbb{R}^d} Q_W(f)(v)\, f^{-1}(v)\, dv = \int W(v, v', v'')$$

$$\times\, f(v) f(v') f(v'') \left[f^{-1}(v) - f^{-1}(v') - f^{-1}(v'') \right]^2$$

$$\times\, \delta_{\{v=v'+v''\}}\, \delta_{\{\omega(v)=\omega(v')+\omega(v'')\}}\, dv dv' dv''.$$

Definition of equilibria

Definition: the equilibria of the 3-waves equation are the functions $f > 0$ such that when

$$v = v' + v''$$

and

$$\omega(v) = \omega(v') + \omega(v''),$$

one has

$$f^{-1}(v) = f^{-1}(v') + f^{-1}(v''),$$

or equivalently, for $g = f^{-1}$,

$$g(v) = g(v') + g(v'').$$

Second part of H-theorem: explicit form of equilibria

It is clear that for all $c \in \mathbb{R}$, $b \in \mathbb{R}^d$, the function

$$g(v) := b \cdot v + c\, \omega(v)$$

is an equilibrium.

Expected result (Second part of H-theorem): For suitable functions ω, all equilibria (in a suitable functional space) have this form.

Natural functional space for g: weighted L^2.

Rigorous result

Theorem (Breden, LD): Let $d \geq 2$ and $\omega \in C^1(\mathbb{R}^d)$ such that

$$\omega(0) = 0, \qquad \nabla\omega(0) = 0,$$

$$\forall v \neq 0, \qquad \omega(v) > 0, \qquad \nabla\omega(v) \neq 0.$$

Assume also that $\omega^{-1}(\{a\})$ is connected for all $a \in \mathbb{R}$.

Let $g \in C^1(\mathbb{R}^d)$ be an equilibrium.

Then, there exist $c \in \mathbb{R}$ and $b \in \mathbb{R}^d$ such that, for all v in $\mathbb{R}^d$,

$$g(v) = b \cdot v + c\,\omega(v).$$

Method of proof

First step: One first assumes that $\nabla g(0) = 0$. Then

$$\nabla g(v) \; // \; \nabla \omega(v),$$

so that

$$g(v) = \mu(\omega(v))$$

for some μ which is C^1 on $\omega(\mathbb{R}) - \{0\}$ and continuous at point 0.

Second step The function μ is additive on its range:

$$\mu(a+b) = \mu(a) + \mu(b),$$

so that it is in fact linear, and

$$g(v) = c\,\omega(v).$$

Third step Finally, one considers $v \mapsto g(v) - \nabla g(0)\, v$ in order to conclude.

Counterexamples

Empty assumptions: For $\omega(v) = |v|^\alpha$ with $\alpha \in]0,1[$, it is not possible to find (nontrivial) $v = v' + v''$ such that $\omega(v) = \omega(v') + \omega(v'')$.

Borderline case: For $\omega(v) = |v|$ in dimension 2, all $g(r,\theta) = r\,h(\theta)$ are equilibria.

One strange case: For $\omega(v) = \frac{v_1}{1+|v|^2}$ in dimension 2, the function $g(v) = \arctan\left(\frac{v_1\sqrt{3}+v_2}{|v|^2}\right) - \arctan\left(\frac{-v_1\sqrt{3}+v_2}{|v|^2}\right)$ is an equilibrium.

Perspectives

- Better result for the 3-waves equation (current result far from optimal!)
- Spectral gaps for the 3-waves and 4-waves linearized equations