Half-Space Problems for the Boltzmann Equation with Phase Transition at the Boundary

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Workshop “Recent Advances in Kinetic Theory and Applications”
Rome, INDAM, November 15th 2019

Work with N. Bernhoff
arXiv:1909.00034[math.AP]

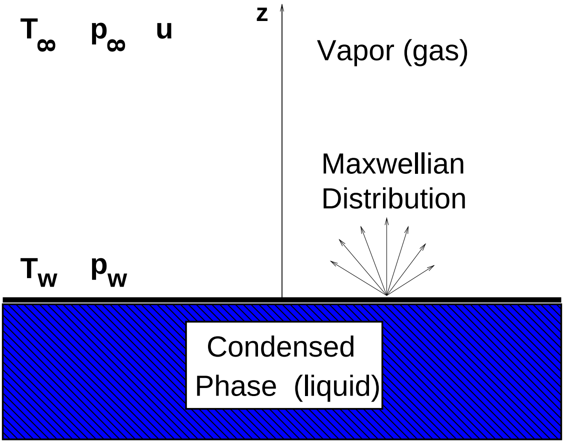

Figure: Interface liquid temperature T_w, saturating vapor pressure p_w
As $z \to +\infty$, the gas distribution function converges to a Maxwellian with temperature T_∞, pressure p_∞ and bulk velocity $(0, 0, u)$

Sone's Half-Space Problem

Unknown distribution function $F \equiv F(z, v)$ satisfying

$$\begin{cases} v_z \partial_z F(z,v) = \mathcal{C}(F)(z,v)\,, & z > 0\,,\ \ v \in \mathbf{R}^3 \\ F(0,v) = \mathcal{M}_{p_w,0,T_w}(v)\,, & v_z > 0 \\ F(z,v) \to \mathcal{M}_{p_\infty,u,T_\infty}(v)\,, & z \to +\infty \end{cases}$$

Maxwellians

$$\mathcal{M}_{p,u,T}(v) := \frac{p}{(2\pi)^{3/2} T^{5/2}} \exp\left(-\frac{v_x^2 + v_y^2 + (v_z - u)^2}{2T} \right)$$

Boltzmann collision integral for hard spheres denoted $\mathcal{C}(F)(z, v)$

$$\iint_{\mathbf{R}^3 \times \mathbf{S}^2} (F(z,v')F(z,v'_*) - F(z,v)F(z,v_*))|(v - v_*) \cdot \omega| dv_* d\omega$$

where $\quad v' := v - (v - v_*) \cdot \omega\omega\,, \qquad v'_* := v_* + (v - v_*) \cdot \omega\omega$

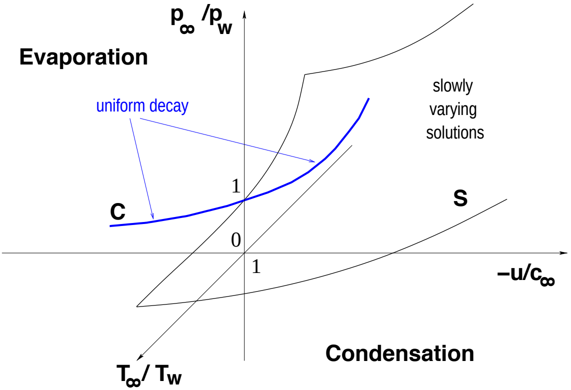

Figure: A solution to Sone's half-space problem exists iff the parameters $(T_\infty/T_w, -u/c_\infty, p_\infty/p_w)$ lie on the curve C in the evaporation case $(0 < u \ll 1)$ and, in the condensation case $(0 < -u \ll 1)$, lie on the surface S. Here $c_\infty := \sqrt{\frac{5}{3}T_\infty}$ = speed of sound as $z \to +\infty$

Perturbation Setting

Smallness assumption $|p_\infty/p_w - 1| + |u/c_\infty| + |T_\infty/T_w - 1| \ll 1$

WLOG assume $p_\infty = T_\infty = 1$ and set $M(\xi) := \mathcal{M}_{1,0,1}(\xi)$;

$$F(z,v) := M(\xi)(1 + f(z,\xi))\,, \quad \xi := v - (0,0,u)$$

with symmetry $f(z,\xi_x,\xi_y,\xi_z) = f(z,-\xi_x,-\xi_y,\xi_z)$

Sone's half-space problem becomes

$$\begin{cases} (\xi_z + u)\partial_z f(z,v) + \mathcal{L}f(z,\xi) = \mathcal{Q}(f)(z,\xi)\,, & z > 0\,, \quad \xi \in \mathbf{R}^3 \\ f(0,\xi) = \frac{\mathcal{M}_{p_w,-u,T_w}(\xi)}{M(\xi)} - 1 \quad \text{for } \xi_z > -u\,, & \lim_{z\to\infty} f(z,\xi) = 0 \end{cases}$$

with the notation

$$\mathcal{L}f = -M^{-1}D\mathcal{C}(M)\cdot(Mf)\,, \quad \mathcal{Q}(f) = M^{-1}\mathcal{C}(Mf)$$

Thm There exists $\epsilon, E, \gamma^* > 0$ such that, for all p_w, u, T_w satisfying

$$|p_w - 1| + |u| + |T_w - 1| < \epsilon$$

the half-space problem has a unique solution f_u that is even in ξ_x, ξ_y and decays exponentially as $z \to +\infty$ uniformly in $|u| < \epsilon$

$$\|(1 + |\xi|)^3 \sqrt{M} f_u(z, \cdot)\|_{L^\infty_\xi} \leq E e^{-\gamma z} \quad \text{for all } 0 < \gamma < \gamma^*$$

iff $f_b := \mathcal{M}_{p_w, -u, T_w} / M - 1$ satisfies the two compatibility conditions

$$\int_{\mathbf{R}^3} (\xi_z + u) Y_1[u](\xi) \mathfrak{R}_u[f_b](\xi) M(\xi) d\xi = 0$$

$$\int_{\mathbf{R}^3} (\xi_z + u) Y_2[u](\xi) \mathfrak{R}_u[f_b](\xi) M(\xi) d\xi = 0$$

Notation $\mathfrak{H} := \{f \in L^2(Md\xi) \text{ even in } \xi_x, \xi_y\}$ and $\langle\phi\rangle := \int_{\mathbf{R}^3} \phi M d\xi$

Lemma [Hilbert 1912] The operator $\mathcal{L}$ is self-adjoint, nonnegative and Fredholm on $L^2(Md\xi)$ with

$$\mathrm{Dom}(\mathcal{L}) = L^2((1+|\xi|)Md\xi)\,, \quad \mathrm{Ker}\,\mathcal{L} \cap \mathfrak{H} = \mathrm{span}\{1, \xi_z, |\xi|^2\}$$

$\mathfrak{H}$-orthonormal basis of $\mathrm{Ker}\,\mathcal{L} \cap \mathfrak{H}$, orthogonal for $(\phi\psi) \mapsto \langle\xi_z\phi\psi\rangle$

$$X_\pm \equiv \frac{|\xi|^2 \pm \sqrt{15}\xi_z}{\sqrt{30}}\,, \quad X_0 \equiv \frac{|\xi|^2 - 5}{\sqrt{10}}\,, \quad \langle\xi_z X_\pm^2\rangle = \pm\sqrt{\tfrac{5}{3}}\,, \quad \langle\xi_z X_0\rangle = 0$$

Bardos-Caflisch-Nicolaenko spectral gap for some $\kappa_0 > 0$

$$g \in \mathrm{Dom}(\mathcal{L}) \cap (\mathrm{Ker}\,\mathcal{L})^\perp \implies \langle g\mathcal{L}g\rangle \geq \kappa_0\langle(1+|\xi|)g^2\rangle$$

The Nicolaenko-Thurber Generalized Eigenvalue Problem

NT-GEPbm find $\phi_u \in \mathfrak{H} \cap \mathrm{Dom}(\mathcal{L})$ so that

$$\mathcal{L}\phi_u(\xi) = \tau_u(\xi_z + u)\phi_u \,, \quad \langle(\xi_z + u)\phi_u^2\rangle = -u$$

Prop There exists $r > 0$ and a C^ω map of solns to the NT-GEPbm

$$(-r, r) \ni u \mapsto (\tau_u, \phi_u) \in \mathbf{R} \times (\mathfrak{H} \cap \mathrm{Dom}(\mathcal{L}))$$
$$\sup_{|u|<r} \|(1 + |\xi|)^s \sqrt{M}\phi_u\|_{L^\infty_\xi} \le C_s < \infty$$

One has $0 < |u| < r \implies u\tau_u < 0$ and moreover

$$\phi_u = X_0 + u\psi_u \,, \quad \tau_u = u\dot{\tau}_0 + O(u^2) \text{ with } \dot{\tau}_0 < 0$$

A Good Reason for Studying ϕ_u...

The function $\Phi_u(z,\xi) := e^{-\tau_u z}\phi_u(\xi)$ solves Sone's linearized pbm

$$(\xi_z + u)\partial_z \Phi_u(z,\xi) + \mathcal{L}\Phi(z,\xi) = 0$$

and

$$\underbrace{0 < -u \ll 1}_{\text{condensation}} \implies \underbrace{\Phi_u(x,\xi) = O(\exp(-\tfrac{1}{2}|u||\dot{\tau}_0|z)}_{\text{exponentially small} \implies \textit{admissible}}$$

$$\underbrace{0 < +u \ll 1}_{\text{evaporation}} \implies \underbrace{\exp(+\tfrac{1}{2}u|\dot{\tau}_0|z) = O(\Phi_u(x,\xi))}_{\text{unbounded} \implies \textbf{not admissible}}$$

Conclusion NT-GEPbm defines a smooth branch of slowly varying (i.e. depending on $\zeta = |u|z$) solutions to the linearized Boltzmann equation admissible only for $u < 0$ (i.e. in the condensation case)

Step 1: Penalize $\mathcal{L}$ [Ukai-Yang-Yu CMP2003]

For $\alpha, \beta, \gamma > 0$, define the penalized, linearized collision integral

$$\mathcal{L}^p g := \mathcal{L}g + \alpha\langle(\xi_z + u)gX_+\rangle X_+ - \beta\langle(\xi_z + u)\psi_u g\rangle\phi_u - \gamma(\xi_z + u)g$$

Then f solves Sone's pbm with exponential decay $\gamma > |u\tau_u| > 0$ iff

$$e^{\gamma z}f(z,\xi) \equiv g(z,\xi) - h(z)\phi_u(\xi)$$

satisfies

$$\begin{cases} (\xi_z+u)\partial_z g + \mathcal{L}^p g = e^{-\gamma z}(Q + \langle\phi_u Q\rangle(\xi_z+u)\psi_u) \\ h(z) = -e^{-\gamma z}\int_0^\infty e^{(\tau_u - 2\gamma)y}\langle\psi_u Q\rangle(z+y)dy \end{cases}$$

with

$$Q(z,\xi) = \mathcal{Q}(g(z,\xi) - h(z)\phi_u(\xi))$$
$$\langle(\xi_z+u)gX_+\rangle\rangle = \langle(\xi_z+u)\psi_u g\rangle = 0$$

Observe that

$$\frac{d}{dz}\begin{pmatrix}A_+\\A_0\\B\end{pmatrix}+(\mathcal{A}_u-\gamma I)\begin{pmatrix}A_+\\A_0\\B\end{pmatrix}=0 \qquad \begin{pmatrix}A_+\\A_0\\B\end{pmatrix}:=\begin{pmatrix}\langle(\xi_z+u)X_+g_\gamma\rangle\\\langle(\xi_z+u)X_0g_\gamma\rangle\\\langle(\xi_z+u)\psi_u g_\gamma\rangle\end{pmatrix}$$

For $|u| < r' < r$, the spectrum of $\mathcal{A}_u$ satisfies

$$\lambda_1(u) > \lambda_2(u) > 0 > \lambda_3(u)\,, \qquad \inf_{0<|u|<r'}\lambda_2(u) > 0 > \sup_{0<|u|<r'}\lambda_3(u)$$

Let $u \mapsto (E_1(u), E_2(u), E_3(u))$ be a C^ω basis of $\mathbf{R}^3$ s.t.

$$\mathcal{A}_u^T E_k(u) = \lambda_k(u)E_k(u)\,, \quad k = 1,2,3\,, \quad 0 < |u| < r'$$

Since $\lambda_3(u) < \gamma$, then

$$L^\infty(\mathbf{R}_+) \ni (A_+, A_0, B)(z)^T \cdot E_3 = (A_+, A_0, B)(0)^T \cdot E_3\, e^{(\gamma-\lambda_3)z}$$

$$\implies (A_+, A_0, B)(0)^T \cdot E_3 = 0 = (A_+, A_0, B)(z)^T \cdot E_3$$

Step 3: Removing the penalization

Set

$$Y_j[u](\xi) := E_j(u)^T \cdot (X_+(\xi), X_0(\xi), \psi_u(\xi)), \quad j = 1, 2$$

Choosing γ so that $\lambda_2(u) > \gamma > 0$ for $0 < |u| < r'$, one has

$$\begin{aligned} 0 =& (A_+, A_0, B)(z)^T \cdot E_j = \langle (\xi_z + u) Y_j[u] g \rangle (0) e^{(\gamma - \lambda_j) z} \\ \iff & \langle (\xi_z + u) Y_j[u] g \rangle (0) = 0 \qquad \text{for } j = 1, 2 \end{aligned}$$

Step 4: Why penalizing $\mathcal{L}$?

Lemma There exists $R, \Gamma, \kappa_1 > 0$ s.t. for all $0 < \alpha = \beta = 2\gamma < 2\Gamma$ and all $|u| < R$, the penalized linearized collision integral

$$\mathcal{L}^p g := \mathcal{L}g + \alpha\langle(\xi_z + u)gX_+\rangle X_+ - \beta\langle(\xi_z + u)\psi_u g\rangle\phi_u - \gamma(\xi_z + u)g$$

satisfies $\quad g \in \mathrm{Dom}(\mathcal{L}) \cap \mathfrak{H} \implies \langle g\mathcal{L}^p g\rangle \geq \kappa_1\gamma\langle(1 + |\xi|)g^2\rangle$

With this lemma, one solves the penalized $1/2$-space problem in the near M-equilibrium regime for $|u| \ll 1$, for **ALL** (small) data f_b

This defines $g(0, \cdot) := \mathfrak{R}_u[f_b]$ uniquely. The compatibility condition to remove the penalization is then

$$\langle(\xi_z + u)Y_j[u]\mathfrak{R}_u[f_b]\rangle(0) = 0 \qquad \text{for } j = 1, 2$$

Conclusion

•Thm confirms numerical results obtained in the Kyoto school (Sone [1978], Sone+Aoki, Doi, Ohwada, Sugimoto, Takata 1980-2000)

•Stronger result (including proof of positivity of F) obtained earlier by T.-P. Liu-S.-H. Yu [ARMA2009], using the Green function for the linearized Boltzmann equation

•Thm above uses only classical energy estimates with filtering of slowly varying modes based on the Nicolaenko-Thurber GEPbm

•Do the compatibility conditions so obtained define a C^1 curve — i.e. does the Implicit Function Theorem apply?