The scaling hypothesis for bounded perturbations of the Smoluchowski coagulation equation

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The Smoluchowski coagulation equation

The Smoluchowski equation

$$\partial_t f(t,x) = \frac{1}{2}\int_0^x f(t,x-y)f(t,y)K(y,x-y)\,\mathrm{d}y - \int_0^\infty f(t,x)f(t,y)K(x,y)\,\mathrm{d}y.$$

$f(t,x)$: density of clusters with size $x > 0$ at time $t \geq 0$.

$K(x,y) \geq 0$: coagulation rate of clusters of sizes x and y, symmetric in x, y. **We always assume $K(x,y)$ bounded for our results.**

Shorter: $\partial_t f = C(f,f)$.

Mass $= \int_0^\infty x\, f(t,x)\,\mathrm{d}x$ is conserved.

Long-time behaviour

$$\text{Moments:} \qquad M_k := \int_0^\infty x^k f(t,x)\,\mathrm{d}x.$$

1. M_k with $k > 1$ is increasing;
2. M_k with $k < 1$ is decreasing.

Solutions converge to 0, since they concentrate in larger and larger sizes. **Is there a universal behaviour?**

If $K(x, y)$ is homogeneous of degree λ, a suitable rescaling gives

$$\partial_t g = 2g + x\partial_x g + C(g, g)$$

Self-similar Smoluchowski equation

(Similar as for the heat equation / Fokker-Planck.)

$$\partial_t g = 2g + x\partial_x g + C(g,g)$$

Self-similar Smoluchowski equation

- The cases $K(x,y) = 1$ or $x + y$ or xy are explicitly solvable **[Menon & Pego]**
- Existence of self-similar profiles **[Escobedo & Mischler]**, **[Fournier & Laurençot]**
- Exponential convergence to self-similarity in explicit cases **[Cañizo, Mischler & Mouhot (2010)]**, **[Srinivasan (2011)]**.
- Infinite-mass profiles **[Niethammer, Velázquez]**
- Uniqueness of profiles **[Laurençot]**, **[Niethammer, Throm & Velázquez]**

Can we prove universal behaviour? — **Scaling hypothesis**

Similar problems

The underlying problem is *universal behaviour out of equilibrium*. We may use these strategies:

1. Explicit solutions
2. Entropies
3. Perturbation arguments

Several settings:

1. Smoluchowski [Menon & Pego], [C., Mischler & Mouhot]
2. Inelastic Boltzmann
   [Mischler & Mouhot], [& Rodríguez Ricard],
   [Carrillo & Toscani]
3. Boltzmann + diffusion [Mischler & Mouhot],
   Inelastic Boltzmann + background [Bisi, C. & Lods (2011)],
   [C. & Lods (2016)]
4. Boltzmann + thermal reservoirs at boundary
   [Carlen, Esposito, Lebowitz, Marra & Mouhot]

Assumptions

$$K(x,y) = 2 + \epsilon W(x,y).$$

$$|W(x,y)| \leq 1, \qquad \text{homogeneous of degree } 0.$$

Theorem (C. & Throm, 2019)

if $M_k(0) < +\infty$ for some $k > 2$ and $\epsilon > 0$ small enough,

1. *there is a unique self-similar profile G_ϵ for the kernel K,*
2. *and for all solutions with mass 1 we have*

$$\|g(t,\cdot) - G_\epsilon\|_{L^1_k} \leq Ce^{-\lambda t}\|g_0 - G_\epsilon\|_{L^1_k},$$

where $C \geq 1$ depends only on $\|g_0 - G_\epsilon\|_{L^1_k}$.

Main strategy

In order to carry out the perturbation around the constant kernel, we need:

1. A good understanding of the limiting case.
2. A good understanding of the linearised operator, in a suitable space.
3. Continuity with respect to ϵ, in some sense.

Notice that the perturbed coagulation operator

$$C_\epsilon(f,f) := 2 + \epsilon C_W(f,f)$$

is a bounded perturbation of the constant kernel, in the norms L^1_k:

$$\|f\|_{L^1_k} := \int_0^\infty (1+x)^k |f(x)| \, \mathrm{d}x.$$

Understanding the constant case

The following was proved in [C., Mischler & Mouhot], with some further details in the recent paper with Throm:

Lemma

Solutions g to the the self-similar Smoluchowski equation with constant kernel $a \equiv 2$ satisfy

$$\|g - G\|_2 \leq C e^{-\frac{1}{2}t},$$

where C depends only on $\|g_0\|_2$, $\|g_0\|_{L^1_2}$.

There are also exponential convergence results by **Srinivasan** in $W^{-1,\infty}$ norms.

Which norm do we need to use?

1. We need fast convergence to equilibrium of the constant kernel case in this norm.
2. We need a spectral gap in this norm for the linearised operator.
3. We need an estimate like

$$\|C(f,f)\| \leq \|f\|^{1+\theta} \|f\|_*^{1-\theta},$$

with a norm $\|f\|_*$ that can be bounded *uniformly in time*. The easiest is

$$\|C_K(f,g)\|_{L^1_k} \leq \frac{3}{2} \|K\|_\infty \|f\|_{L^1_k} \|g\|_{L^1_k}$$

This is where restrictions come from!

The linearised operator

$$Lh := 2h + x\partial_x h + C(h, G) + C(G, h).$$

Theorem

This operator has a spectral gap in all spaces L^1_k with $k > 2$:

$$\|h(t, \cdot)\|_{L^1_k} \leq Ce^{-\lambda t}\|h_0\|_{L^1_k} \qquad \text{for all } h_0 \text{ with } \int x h_0 = 0 \text{ (mass 0)}.$$

In [C. Mischler & Mouhot] we proved a spectral gap in $H^{-1}(e^{\mu x})$.

In order to obtain the theorem we can use the spectral gap extension / restriction results in [Gualdani, Mischler & Mouhot].

Sketch of proof: exponential convergence

1. First, show that the linearised operator L_ϵ has a spectral gap. Easy for a bounded perturbation.
2. Show exponential convergence to equilibrium, locally.
3. We can then use the global behaviour of the constant kernel case to extend this result to arbitrarily large regions.

Write the equation with a Taylor expansion of $C(g,g)$:

$$\partial_t g = \partial_t(g - G_\epsilon) = L_\epsilon(g - G_\epsilon) + C_\epsilon(g - G_\epsilon, g - G_\epsilon)$$

Call $h := g - G_\epsilon$. By Duhamel / variation of constants:

$$h_t = e^{tL_\epsilon} h_0 + \int_0^t e^{(t-s)L} C_\epsilon(h_s, h_s)\,\mathrm{d}s.$$

Then

$$\begin{aligned} \|h_t\| \leq & Ce^{-\lambda t}\|h_0\| + \int_0^t e^{-\lambda(t-s)} \|C_\epsilon(h_s, h_s)\|\,\mathrm{d}s \\ \leq & Ce^{-\lambda t}\|h_0\| + K \int_0^t e^{-\lambda(t-s)} \|h_s\|^2\,\mathrm{d}s, \end{aligned}$$

hence convergence happens locally around G.

Thanks for listening!