Approximate controllability of hypoelliptic equations

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Recent advances in kinetic equations and applications
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Usual Riemannian setting:

Main example:

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- $\mathcal{M}$ compact connected manifold,

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- $\mathcal{M} = \mathbb{T}^d$

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What if g vanishes at some points, in some directions?

Typical unique continuation results: Riemannian setting

Theorem (Holmgren, Carleman, Calderón)

An eigenfunction φ_j of Δ_g never vanishes identically on an open set $\omega \neq \emptyset$.

Theorem (Donnelly-Fefferman 1988, Lebeau-Robbiano 95)

Assume $\omega \subset \mathcal{M}$, $\omega \neq \emptyset$. Then $\|\varphi_j\|_{L^2(\mathcal{M})} \leq Ce^{\kappa\sqrt{\lambda_j}} \|\varphi_j\|_{L^2(\omega)}$

- $\rightsquigarrow$ $\|\varphi_j\|_{L^2(\omega)} \gtrsim e^{-\kappa\sqrt{\lambda_j}}$ for normalized eigenfunctions.
- $\rightsquigarrow$ Optimal in general.

Relax the ellipticity condition $g^{ij}\xi_i\xi_j \geq c_0|\xi|^2$, with $c_0 > 0$?

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Typical approximate controllability result: Riemannian setting

Heat equation controlled from ω:

$$\begin{cases} (\partial_t - \Delta_g)u = \mathbb{1}_\omega f, & \text{in } (0,T)\times\mathcal{M}, \\ u(0) = 0, & \text{in } \mathcal{M}. \end{cases} \tag{1}$$

- Exact controllability: find f so that $u(T) = u_1 \in L^2$?

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Theorem (Fernández-Cara-Zuazua 2000, Phung 2004)

Fix $T > 0$*. For any* $\varepsilon > 0$, $u_1 \in L^2(\mathcal{M})$*, there is* $f \in L^2((0,T)\times\omega)$

s.t. the solution of (1) *satisfies*

$$\|u(T) - u_1\|_{H^{-1}(\mathcal{M})} \leq \varepsilon \|u_1\|_{L^2(\mathcal{M})}.$$

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Sub-Riemannian/hypoelliptic setting

- $\mathcal{M}$ compact connected manifold
- ds a density on $\mathcal{M}$, $L^2 = L^2(\mathcal{M}, ds)$
- m vector fields $X_1, \cdots, X_m$
- Type I Hörmander operator

$$\mathcal{L} = \sum_{i=1}^{m} X_i^* X_i.$$

Here $\int_{\mathcal{M}} X^*(u) v \, ds = \int_{\mathcal{M}} u X(v) \, ds \quad \leftrightsquigarrow \quad X^* = -X - \mathrm{div}_{ds}(X)$

- Formally symmetric nonnegative, $\mathcal{L} = -\mathrm{div}_{ds}(\nabla_{SR} \cdot)$

Examples in dimension $d = 2$, $\mathcal{M} = \mathbb{T}^2 = [-1, 1)^2$, $ds = dx_1 dx_2$:

- Elliptic operator: $X_1 = \partial_{x_1}$, $X_2 = \partial_{x_2} \implies \mathcal{L} = -(\partial_{x_1}^2 + \partial_{x_2}^2)$ is elliptic.
- Grushin operator:

$$X_1 = \partial_{x_1}, \quad X_2 = x_1 \partial_{x_2} \implies \quad \mathcal{L} = -\left(\partial_{x_1}^2 + x_1^2 \partial_{x_2}^2\right)$$

- p-Grushin operators:

$$X_1 = \partial_{x_1}, \quad X_2 = x_1^p \partial_{x_2} \implies \quad \mathcal{L}_p = -\left(\partial_{x_1}^2 + x_1^{2p} \partial_{x_2}^2\right)$$

Definition

with $\mathscr{F} = (X_1, \cdots, X_m)$ set $\mathrm{Lie}^\ell(\mathscr{F})$:

- $\mathrm{Lie}^1(\mathscr{F})(x) = \mathrm{span}(X_1(x), \cdots, X_m(x))$,
- $\mathrm{Lie}^{\ell+1}(\mathscr{F}) = \mathrm{span}\left(\mathrm{Lie}^\ell(\mathscr{F}) \cup \left\{[X, X_j]; X \in \mathrm{Lie}^\ell(\mathcal{F}), j = 1, \cdots, m\right\}\right)$.

Assumption (Chow-Rashevski-Hörmander)

- *$\exists \ell \geq 1$ so that for any $x \in \mathcal{M}$, $\mathrm{Lie}^\ell(X_1, \cdots, X_m)(x) = T_x\mathcal{M}$.*
- *set $k :=$ the minimal ℓ.*

Examples:

- Elliptic operator: $X_1 = \partial_{x_1}$ and $X_2 = \partial_{x_2}$ $\rightsquigarrow k = 1$
- Grushin operator: $X_1 = \partial_{x_1}$ and $X_2 = x_1\partial_{x_2}$ $\rightsquigarrow k = 2$
  since $[\partial_{x_1}, x_1\partial_{x_2}] = \partial_{x_2}$
- p-Grushin operators: $X_1 = \partial_{x_1}$ and $X_2 = x_1^p\partial_{x_2}$ $\rightsquigarrow k = p + 1$

Theorem (Chow-Rashevski, 1938)

Assume Chow-Rashevski-Hörmander condition. For any $x_0, x_1 \in \mathcal{M}$, there is a curve $[0,1] \to \mathcal{M}$, $t \mapsto \gamma(t)$ such that

- $\gamma(0) = x_0$ *and* $\gamma(1) = x_1$
- γ *is always tangent to* $\mathrm{span}(X_1, \cdots, X_m)$

Theorem (Hörmander 1967, Rothschild-Stein 1976)

Assume Chow-Rashevski-Hörmander condition.

- *The operator* $\mathcal{L}$ *is hypoelliptic:* $\forall u \in \mathscr{D}'(\mathcal{M}), x_0 \in \mathcal{M}$

$$\mathcal{L}u \in C^\infty \text{ near } x_0 \implies u \in C^\infty \text{ near } x_0.$$

- *The operator* $\mathcal{L}$ *is subelliptic of order* $\frac{1}{k}$*:*

$$\|u\|_{H^{\frac{2}{k}}(\mathcal{M})} \lesssim \|\mathcal{L}u\|_{L^2(\mathcal{M})} + \|u\|_{L^2(\mathcal{M})}$$

Examples:

- Elliptic operators $\rightsquigarrow k = 1$: $\|u\|_{H^2(\mathcal{M})} \lesssim \|\mathcal{L}u\|_{L^2(\mathcal{M})} + \|u\|_{L^2(\mathcal{M})}$
- Grushin operator $\rightsquigarrow k = 2$: $\|u\|_{H^1(\mathcal{M})} \lesssim \|\mathcal{L}u\|_{L^2(\mathcal{M})} + \|u\|_{L^2(\mathcal{M})}$
- p-Grushin operators $\mathcal{L}_p = -\left(\partial_{x_1}^2 + x_1^{2p}\partial_{x_2}^2\right) \rightsquigarrow k = p+1$
  $\|u\|_{H^{\frac{2}{p+1}}(\mathcal{M})} \lesssim \|\mathcal{L}u\|_{L^2(\mathcal{M})} + \|u\|_{L^2(\mathcal{M})}$

Properties of $\mathcal{L}$:

$$\mathcal{L} : D(\mathcal{L}) \subset L^2(\mathcal{M}) \to L^2(\mathcal{M}),$$

- subelliptic estimates $\Longrightarrow$ $H^2(\mathcal{M}) \subset D(\mathcal{L}) \subset H^{\frac{2}{k}}(\mathcal{M})$

$\rightsquigarrow$ $\mathcal{L}$ is selfadjoint on $L^2(\mathcal{M})$, with compact resolvent

$\rightsquigarrow$ Hilbert basis of eigenfunctions $(\varphi_j)_{j\in\mathbb{N}}$, real eigenvalues $(\lambda_j)_{j\in\mathbb{N}}$

$$\mathcal{L}\varphi_i = \lambda_i\varphi_i, \quad (\varphi_i, \varphi_j)_{L^2(\mathcal{M})} = \delta_{ij}, \quad 0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_j \to +\infty.$$

$\rightsquigarrow$ $\varphi_j \in C^\infty(\mathcal{M})$.

$\rightsquigarrow$ Well-posedness of hypoelliptic wave and heat equations $(\partial_t^2 + \mathcal{L})v = f$ and $(\partial_t + \mathcal{L})u = f$

Assumption (Analyticity)

The manifold $\mathcal{M}$, the density ds, and the vector fields X_i are **real-analytic**.

$\rightsquigarrow$ the Chow-Rashevski-Hörmander is necessary for attainability/hypoellipticity.

Theorem (Bony 1969)

An eigenfunction φ_j of $\mathcal{L}$ never vanishes identically on an open set $\omega \neq \emptyset$.

Theorem

Let $\omega \subset \mathcal{M}$, $\omega \neq \emptyset$. Then, for normalized eigenfunctions:

$$\|\varphi_j\|_{L^2(\omega)} \geq Ce^{-c\lambda_j^{k/2}}$$

- False in general without the analyticity assumption (Bahouri 1986).

Proposition (Csq of Beauchard-Cannarsa-Guglielmi 2017)

For the p-Grushin examples, there are $\omega \neq \emptyset$ and (λ_j, φ_j) eigenvalues/eigenfunctions of $\mathcal{L}_p$ s.t.

$$\|\varphi_j\|_{L^2(\omega)} \leq Ce^{-c_0\lambda_j^{k/2}}, \quad k = p + 1.$$

hypoelliptic heat equation: controllability

Sobolev norms:

$$\|u\|_{\mathcal{H}^s_{\mathcal{L}}} = \left\|(1+\mathcal{L})^{\frac{s}{2}} u\right\|_{L^2(\mathcal{M})}, \quad s \in \mathbb{R}.$$

Hypoelliptic heat equation controlled from ω:

$$\begin{cases} (\partial_t + \mathcal{L})u = \mathbb{1}_\omega f, & \text{in } (0,T)\times\mathcal{M}, \\ u(0) = 0, & \text{in } \mathcal{M}. \end{cases} \tag{2}$$

Approximate controllability: drive the solution to $u(T) \approx u_1$?

Corollary (Approximate controllability and its cost)

Fix $T > 0$. For any $\varepsilon > 0$, $u_1 \in L^2(\mathcal{M})$, there is $f \in L^2((0,T)\times\omega)$

s.t. the solution of (2) satisfies

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$$\|f\|_{L^2((0,T)\times\omega)} \leq C e^{\frac{c}{\varepsilon^k}} \|u_1\|_{L^2(\mathcal{M})},$$

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hypoelliptic heat equation: observability

Hypoelliptic free heat equation:

$$\begin{cases} \partial_t y + \mathcal{L} y = 0, & \text{in } (0,T)\times\mathcal{M}, \\ y(0) = y_0 & \text{in } \mathcal{M}, \end{cases}$$

Theorem (Approximate observability)

For all $T > 0$, there are $C, c > 0$ s.t. for all $y_0 \in \mathcal{H}^1_{\mathcal{L}}$, for all $\varepsilon > 0$

$$\|y_0\|^2_{L^2} \le C e^{\frac{c}{\varepsilon^k}} \int_0^T \|y(t)\|^2_{L^2(\omega)} dt + \varepsilon^2 \|y_0\|^2_{\mathcal{H}^1_{\mathcal{L}}} ,$$

About the proofs

Theorem (Approximate observability)

For all $T > 0$, there are $C, c > 0$ s.t. for all $y_0 \in \mathcal{H}^1_{\mathcal{L}}$, for all μ large

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   $\rightsquigarrow$ Laurent-L. 2015-2019

2. A (sub-Riemmanian) geometric construction
   $\rightsquigarrow$ Rifford-Trélat 2005

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For all $T > 0$, there are $C, c > 0$ s.t. for all $y_0 \in \mathcal{H}^1_{\mathcal{L}}$, for all μ large

$$\|y_0\|^2_{L^2} \leq Ce^{\mu^k} \int_0^T \|e^{-t\mathcal{L}} y_0\|^2_{L^2(\omega)} dt + \frac{1}{\mu^2} \|y_0\|^2_{\mathcal{H}^1_{\mathcal{L}}} ,$$

4 main steps/ingredients:

1. Quantitative Unique Continuation for $\partial_t^2 + \mathcal{L}$ (hypoelliptic wave equation)
   $\rightsquigarrow$ Laurent-L. 2015-2019
2. A (sub-Riemmanian) geometric construction
   $\rightsquigarrow$ Rifford-Trélat 2005
3. Subelliptic estimates (H^s norms $\leftrightsquigarrow$ $\mathcal{H}^s_{\mathcal{L}}$ norms)
   $\rightsquigarrow$ Rotschild-Stein 1976

About the proofs

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4. From $\partial_t^2 + \mathcal{L}$ (waves) to $\mathcal{L} - \lambda_j$ (eigenfunctions) or $\partial_t + \mathcal{L}$ (heat): transmutation
   $\rightsquigarrow$ Ervedoza-Zuazua 2011

About the proofs

Theorem (Approximate observability)

For all $T > 0$, there are $C, c > 0$ s.t. for all $y_0 \in \mathcal{H}^1_{\mathcal{L}}$, for all μ large

$$\|y_0\|_{L^2}^2 \leq Ce^{\mu^k} \int_0^T \|e^{-t\mathcal{L}} y_0\|^2_{L^2(\omega)} dt + \frac{1}{\mu^2} \|y_0\|^2_{\mathcal{H}^1_{\mathcal{L}}},$$

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The proofs: Quantitative Unique Continuation

- Global UC statements $\leftarrow$ local UC results + geometric constructions.

Local (near x^0) UC result across $\{\phi = 0\} \ni x^0$ for $P = p(x, D_x)$:

$$\big(Pu = 0 \text{ near } x^0, \quad u = 0 \text{ in } \{\phi > 0\}\big) \overset{?}{\Longrightarrow} u = 0 \text{ near } x^0.$$

Holmgren-John (1949)

- analytic coefficients
- ϕ non characteristic for P: $p(x^0, d\phi(x^0)) \neq 0$

Carleman-Hörmander (1960)

- C^∞ (even C^1) coefficients
- ϕ pseudoconvex for P: $\{p, \{p, \phi\}\}(x^0, \xi) > 0$

Quantitative Carleman-Hörmander theorem

Usual Hörmander theorem: 3 steps:

1. Carleman estimates:

$$\left\| e^{\tau\psi} v \right\|_{L^2} \lesssim \left\| e^{\tau\psi} P v \right\|_{L^2}, \qquad \text{for all } \tau \geq \tau_0,$$

   v compactly supported near x^0. Here, $\psi =$ convexification of ϕ.

2. Apply it with $v = \chi u$ where $Pu = 0$, $\chi \to$ levelsets of ψ. Yields ($\mu = \tau$)

$$\|u\|_{V_2} \lesssim e^{\kappa\mu} \|u\|_{V_1} + \underbrace{e^{-\kappa'\mu} \|u\|}_{\text{expo. small remainder}}$$

3. propagates very well (Bahouri 87, Robbiano 95, Lebeau-Robbiano 95):

$$\|u\|_{L^2(K)} \lesssim e^{\kappa\mu} \|u\|_{H^1(\tilde{\omega})} + \underbrace{e^{-\kappa'\mu} \|u\|_{H^1}}_{\text{expo. small remainder}}, \qquad Pu = 0.$$

Quantitative Holmgren-John theorem

(Tataru-Robbiano-Zuily-Hörmander spirit)

- A Carleman estimate "localized in $\xi = 0$"

$$\left\| e^{-\frac{\varepsilon}{2\tau}|D|^2} e^{\tau\psi} v \right\| \lesssim \left\| e^{-\frac{\varepsilon}{2\tau}|D|^2} e^{\tau\psi} P v \right\| + e^{-\tau \mathsf{d}} \left\| e^{\tau\psi} v \right\|, \quad \tau \geq \tau_0$$

- Apply it with $v = \chi u$, $\chi \to$ levelsets of ψ. Yields ($Pu = 0$) $\left\| e^{-\frac{\varepsilon}{2\tau}|D|^2} e^{\tau\psi} \chi u \right\| \lesssim e^{\kappa\tau} \|u\|_{V_1} + e^{-\delta\tau} \|u\|$ for all $\tau \geq \tau_0$.
- Complex analysis in the τ variable $\rightsquigarrow$ Local estimate

$$\|u\|_{V_2} \leq e^{\kappa\mu} \|u\|_{V_1} + \underbrace{\frac{C}{\mu} \|u\|}_{\text{not so small remainder}} .$$

PROBLEM: does not propagate well $\rightsquigarrow e^{e^{e^{e^{\cdots e^{\mu}}}}}$

- Solution! propagate low frequencies only: with $m \in C_c^\infty(\mathbb{R})$:

$$\left\| m\left(\frac{|D|}{\mu}\right) \chi_{V_2} u \right\| \leq C e^{\kappa\mu} \left\| m\left(\frac{|D|}{\mu}\right) \chi_{V_1} u \right\| + C \underbrace{e^{-\kappa'\mu} \|u\|}_{\text{expo. small remainder}} ,$$

for all $\mu \geq \mu_0$ and $u \in C_c^\infty(\mathbb{R}^n)$.

- PROBLEM: Commutators $\left[m\left(\frac{|D|}{\mu}\right), \chi(x) \right]$ are of order $\mu^{-\infty} \to$ too bad
- Solution! analytic cutoff functions!

Thank you

THANK YOU FOR YOUR ATTENTION!