The diffusive limit for Carleman-type kinetic models

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Abstract

We study the limiting behavior of the Cauchy problem for a class of Carleman-like models in the diffusive scaling with data in the spaces L^p, $1 \le p \le \infty$. We show that, in the limit, the solution of such models converges towards the solution of a nonlinear diffusion equation with initial values determined by the data of the hyperbolic system. When the data belong to L^1, a condition of conservation of mass is needed to uniquely identify the solution in some cases, whereas the solution may disappear in the limit in other cases.

1 Introduction

We consider a class of one-dimensional models for a gas composed of two kinds of particles moving parallel to the x-axis with constant and equal speeds, of modulus $c > 0$, one in the positive x-direction with density $u(x,t)$, the other in the negative x-direction with density $v(x,t)$. A general version of such a model has the following form:

$$\text{(1)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} + c\dfrac{\partial u}{\partial x} = k(u,v,x)(v-u) \\[2ex] \dfrac{\partial v}{\partial t} - c\dfrac{\partial v}{\partial x} = k(u,v,x)(u-v), \end{cases}$$

where $u = u(x,t)$, $v = v(x,t)$, $x \in \Omega \subseteq \mathbb{R}$, $t \ge 0$, and $k(u,v,x)$ is a nonnegative function, called the interaction rate (also rate function or rate coefficient), that characterizes the interactions between gas particles. The model is in local equilibrium when $u = v$, a situation that will be obtained in the limit as we will see.

We devote our main effort to the case $\Omega = \mathbb{R}$ with some comments on the application to bounded Ω with Neumann (specular) boundary conditions.

The case $k(u,v,x) = u+v$ was introduced by Carleman in the 1930's as a simplified model of the Boltzmann equation [3], and has been subsequently studied by many authors (for a survey on the mathematical theory of these models see, for example, [16]).

The variables can be easily scaled in such a way that (1) is reduced to the form

$$\text{(2)} \qquad \begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} + \dfrac{1}{\varepsilon}\dfrac{\partial u_\varepsilon}{\partial x} = \dfrac{1}{\varepsilon^2}k(u_\varepsilon, v_\varepsilon, x)(v_\varepsilon - u_\varepsilon) \\[2ex] \dfrac{\partial v_\varepsilon}{\partial t} - \dfrac{1}{\varepsilon}\dfrac{\partial v_\varepsilon}{\partial x} = \dfrac{1}{\varepsilon^2}k(u_\varepsilon, v_\varepsilon, x)(u_\varepsilon - v_\varepsilon) \qquad x \in \Omega \subseteq \mathbb{R},\ t \ge 0, \end{cases}$$

with parameter $\varepsilon > 0$. This scaling is particularly interesting because the limit $\varepsilon \to 0^+$ (called *hydrodynamical limit*) leads, at least formally, to diffusive type equations which can be viewed as the Navier-Stokes equations of the ficticious gas. Hence the name of *diffusive scaling.* Note that the speed $1/\varepsilon \to \infty$.

The process is as follows: we introduce two macroscopic variables, the mass density ρ_ε and the flux j_ε defined by

$$\rho_\varepsilon = u_\varepsilon + v_\varepsilon, \qquad j_\varepsilon = \frac{1}{\varepsilon}\,(u_\varepsilon - v_\varepsilon). \tag{3}$$

In the typical case when the rate has the form $k_\alpha(u_\varepsilon, v_\varepsilon, x) = (u_\varepsilon + v_\varepsilon)^\alpha$, most considered in the literature, System (2) is equivalent to the following macroscopic equations for the mass density and the flux

$$\begin{cases} \dfrac{\partial \rho_\varepsilon}{\partial t} + \dfrac{\partial j_\varepsilon}{\partial x} = 0 \\[2ex] \varepsilon^2 \dfrac{\partial j_\varepsilon}{\partial t} + \dfrac{\partial \rho_\varepsilon}{\partial x} = -2\rho_\varepsilon^\alpha j_\varepsilon, \end{cases} \tag{4}$$

posed in $(x,t) \in \Omega \times (0,T)$ with initial data for density and flux, $\rho_\varepsilon(x,0) = u_0(x) + v_0(x)$ and $j_\varepsilon(x,0) = (u_0(x) - v_0(x))/\varepsilon$. If we are now allowed to disregard the term $\varepsilon^2\,\partial j_\varepsilon/\partial t$ in the limit $\varepsilon \to 0$, we formally obtain the following nonlinear heat equation for the limit density $\rho = \lim_{\varepsilon\to 0} \rho_\varepsilon$:

$$\frac{\partial \rho}{\partial t} = \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{1}{\rho^\alpha}\frac{\partial \rho}{\partial x}\right), \tag{5}$$

with initial conditions $\rho(x,0) = u_0(x) + v_0(x)$. This is called the *diffusive limit*. Much more general forms of the rate function k can be admitted in the study of this limit process (a set of convenient assumptions on k will be stated and discussed below). For such general k the factor ρ^α in System (4) must be replaced by

$$f_\varepsilon(x,t) = k((\rho_\varepsilon + \varepsilon j_\varepsilon)/2\,,\,(\rho_\varepsilon - \varepsilon j_\varepsilon)/2\,,\,x).$$

Assuming that in the limit $\varepsilon j_\varepsilon \to 0$, the denominator ρ^α in the limit equation (5) becomes $k(\rho/2, \rho/2, x)$.

An interesting mathematical problem is posed, i.e., justifying this limit process for different choices of α (or, more generally, the function k) and under suitable assumptions on the data. It has been the object of a number of papers, which give information on the convergence of $\rho_\varepsilon = u_\varepsilon + v_\varepsilon$ as $\varepsilon \to 0^+$ to a function $\rho(x,t)$, solution of the nonlinear diffusive process.

The study has been done specially in the model cases $k_\alpha(u_\varepsilon, v_\varepsilon, x) = (u_\varepsilon + v_\varepsilon)^\alpha$. When $\alpha = 0$ we obtain in a rather easy way the solutions of the standard diffusive process, the heat equation (the situation is a bit more involved in the case of the initial-boundary value problem, see [20]).

When $\alpha = 1$ (which corresponds to the Carleman model) and under a number of conditions of the data, the asymptotic theory in the whole space has been analysed by Kurtz [9] and McKean [13], whereas Fitzgibbon [7] studied the problem in a bounded domain, with specular boundary conditions. Subsequently, Pulvirenti and Toscani [17] have extended the theory in the case $\alpha \in [0,1)$, in the framework of C^1 data, whereas Lions and Toscani [11] have solved the case $\alpha \in (-\infty, 1)$, with more general integrable data. We are strongly inspired by this work.

Here are the main novelties introduced in our paper: a new mathematical approach allows to weaken the regularity hypotheses on the data; we merely assume that the data u_0 and v_0 belong to the natural space $L^1_+(\mathbb{R})$ without additional, unphysical restrictions; we recover the theory for $\alpha \in [-1, 1)$; we also include the limit case $\alpha = 1$ where the diffusive limit does not have uniqueness of solutions, and we characterize our limit as the unique maximal solution, see Theorem 12. We also explore the supercritical range $\alpha > 1$, and show a new qualitative result: for $\alpha \geq 2$ for integrable data $u_0, v_0 \in L^1(\mathbb{R})$, the hydrodynamical limit is trivial, $\rho = 0$. Finally, the theory for other problems or other types of data is briefly discussed. In particular, for $|\alpha| \leq 1$ a theory is constructed with data in $L^1 + L^\infty(\mathbb{R})$ which includes all spaces $L^p(\mathbb{R})$, $2 \leq p \leq \infty$.

A strategy for very general systems of kind (2) has been proposed by Marcati and Rubino [12]. However, in their paper k is not allowed to vanish and the target equation is not degenerate. We finally mention that another remarkable paper, which studies the same kind of problems, has been very recently written by Donatelli and Marcati [5]. In that paper, the authors study a very general nonlinear pseudo-differential system, in the framework of L^2 initial data. Their results complement but do not overlap our work.

2 Description of the main results of the paper

We want to contribute to the theory of the diffusive limit for the above models. Our main interest is to develop a theory that applies to all nonnegative data in the typical functional spaces $L^p(\Omega)$, $1 \leq p \leq \infty$, and to include interaction rates k of type $k_\alpha(u_\varepsilon, v_\varepsilon, x) = (u_\varepsilon + v_\varepsilon)^\alpha$ for different values of $\alpha \in \mathbb{R}$.

(i) We devote a first effort to develop the theory when k does not vanish (more precisely, for all regular k's, in the sense defined in the next section). This is done in Sections 3 to 5 and culminates in Theorem 3, where we prove the convergence of the solutions $(u_\varepsilon, v_\varepsilon)$ of System (2) to the diffusive limit under the assumptions that the data $u_{0\varepsilon}, v_{0\varepsilon}$ are nonnegative and uniformly bounded (in $L^\infty(\mathbb{R})$), and converge locally weakly to functions u_0, v_0. Actually, we show that the limit $\rho = \lim_{\varepsilon\to 0}(u_\varepsilon + v_\varepsilon)$ solves the nonlinear diffusion equation

(6) $$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x}\left(D(\rho)\frac{\partial \rho}{\partial x}\right),$$

a generalization of (5) with diffusivity $D(\rho) = 1/(2k(\rho/2, \rho/2, x))$. Besides, $\rho(x,t)$ takes initial data $\rho_0 = u_0 + v_0$.

(ii) The main analytical problem of these models lies in the fact that k often vanishes or becomes infinite, as in k_α for $\alpha \neq 0$. The previous approach justifies the diffusive limit even for these rates if the initial data satisfy the conditions $0 < \delta \leq u_0, v_0 \leq M$, since then $\delta \leq u_\varepsilon, v_\varepsilon \leq M$ and k_α is regular on the range of values of the solutions for all α.

For general solutions $u, v \geq 0$ our strategy consists in avoiding the difficulty in a first stage by regularizing the problem, so that k does not vanish on the range of the solutions. This can be done in different ways. The simplest one in our opinion is based on *lifting* the initial data and defining

(7) $$u_{0\delta}(x) = u_0(x) + \delta, \qquad v_{0\delta}(x) = v_0(x) + \delta.$$

We are then faced with the problem of solving System (2) with lifted data. By writing

(8) $$u_\varepsilon = \bar{u}_\varepsilon + \delta, \qquad v_\varepsilon = \bar{v}_\varepsilon + \delta,$$

we transform solving System (2) for $(u_\varepsilon, v_\varepsilon)$ with lifted data $(u_{0\delta}, u_{0\delta})$, into the an equivalent system for $(\bar{u}_\varepsilon, \bar{v}_\varepsilon)$ with initial data u_0, v_0 and rate $\bar{k}$ defined by $\bar{k}(u, v, x) = k(u + \delta, v + \delta, x)$. When applied to the typical $k_\alpha = (u + v)^\alpha$, the change allows to assume the new interaction rate does not vanish. It is shown that $\bar{k}_\alpha = (u + v + 2\delta)^\alpha$ has all the properties required to develop a good theory, both in L^1 and in L^∞. These properties are listed in Definitions 1 and following.

For general data in $L^1(\mathbb{R})$ we still have to pass to the limit $\delta \to 0^+$ to recover the solutions of the original problem. The idea is to pass to the limit along subsequences $\varepsilon \to 0^+$, $\delta \to 0^+$ to obtain a function

$$\rho(x, t) = \lim_{\varepsilon, \delta \to 0^+} \rho_{\varepsilon, \delta}(x, t)$$

which solves the diffusion equation and takes on the initial data. Such a program can be performed for the rates $k = k_\alpha$ under three conditions: (a) the exponent satisfies $\alpha < 2$; (b) the limit is taken along special sequences $(\varepsilon, \delta) \to (0, 0)$ such that δ is not too small with respect to ε; and (c) data belong to $L^1 \cap L^\infty$. The precise result is formulated in Theorems 7 and 8.

The appearance of the exponent $\alpha = 2$ reflects the properties of the target equation. We recall that the diffusive equation with $\alpha > 0$ falls into the class of *fast diffusion equations*. It is known that existence or uniqueness may fail for certain ranges of α. Thus, for $\alpha \geq 2$ the diffusive limit breaks down in the sense that, for data $\rho_0 = u_0 + v_0 \in L^1_+(\mathbb{R})$ the target equation (5) does not admit solutions with finite mass, see [21] and its references. Indeed, the limit of the solutions ρ_δ of the target equation with lifted data vanishes uniformly on sets of the form $t \geq \tau > 0$. We show that the hydrodynamical limit, taken along suitable sequences $(\varepsilon, \delta) \to (0, 0)$, behaves in the same vanishing way. It means that *for $\alpha \geq 2$ the diffusive scaling we are using is too slow and does not allow to see anything significant for large times.*

(iii) A much finer theory can be developed in the range $|\alpha| \leq 1$ where we show that the functional setting corresponds to dissipative operators, see Section 7 and Appendix I. This is a typical property of nonlinear diffusion equations of the types dealt with here, and it implies that the equations generated semigroups of contractions in the corresponding functional space, here $L^1_+(\mathbb{R})$. Applying this concept with data in that space, we prove the convergence result to the diffusive limit eliminating all additional conditions on the data of previous works. The result is precisely stated in Theorem 12, and can be briefly stated as follows:

Theorem A *Let $|\alpha| \leq 1$. The semigroups $S_{\varepsilon,\alpha}$ generated by the hyperbolic system acting on $X = L^1_+(\mathbb{R})$ converge as $\varepsilon \to 0^+$ towards S_α, the semigroup generated by the target equation. Convergence takes place in the typical semigroup topology $C([0, \infty); L^1(\mathbb{R}))$.*

We recall that existence and uniqueness of nonnegative solutions of the Cauchy problem for the target equation is guaranteed for $\alpha < 1$, but it does not hold for $\alpha \geq 1$ if the data are integrable. Indeed, for $1 \leq \alpha < 2$ the problem with (nontrivial) integrable initial data $\rho_0 \in L^1(\mathbb{R})$, $\rho_0 \geq 0$, admits infinitely many smooth solutions $\rho \in C([0, T]; L^1(\mathbb{R})) \cap C^\infty(\mathbb{R} \times [0, T])$, [6, 18, 19]. Our limit selects the maximal one, which exists for all times $t > 0$ and is characterized by the property of mass conservation:

$$\int_{\mathbb{R}} \rho(x, t)\, dx = \int_{\mathbb{R}} \rho_0(x)\, dx \quad \forall t > 0.$$

In our proof we use heavily the fact that the semigroup is order-preserving and L^1-contractive whenever $|\alpha| \leq 1$, mimicking similar properties of the diffusive problem.

In the case $\alpha \in [1, 2)$ the diffusive equation is still contractive for maximal solutions in L^1. However, for $\alpha > 1$ the hyperbolic semigroup is neither contractive nor order-preserving, but only ω-dissipative with a bad dependence of ω on ε (see Appendix I). This is the reason why we are only able to pass to the limit in a very special way for $\alpha > 1$. Further investigation is in progress in this range.

(iv) Extensions: it is also known that the diffusive equation can be solved in the fast diffusion range with data ρ_0 in the much wider class $L^1_{loc}(\mathbb{R})$. In that direction, we first show that the hyperbolic system can be solved in the same class using the concept of mild solutions. We also show that the diffusive limit can be justified for $|\alpha| \le 1$ when the data belong to the class $L^1(\mathbb{R}) + L^\infty(\mathbb{R})$, which includes $L^p(\mathbb{R})$ for all $p \le 1$. This is done as an easy extension of the L^1 and L^∞ theories.

Another interesting variant consists in taking as base space a finite interval, say $I = (-R, R)$. We can solve for System (2) with Neumann conditions $u_x + v_x = 0$ at the lateral boundaries $x = \pm R$, $t > 0$ (also known as mass preserving or specular conditions). We can then re-do the L^∞ theory for regular k's and the L^1 theory for k_α, $|\alpha| \le 1$, with no major changes, thus establishing the diffusive limit of the hyperbolic system.

One immediate application of the last result is the diffusive limit in $\mathbb{R}$ with L^1-periodic initial data.

Outline. The paper is organized as follows: in the next section we collect many preliminary results on the hyperbolic model. Section 3 is then devoted to establish local estimates. We pass to the limit for regular k in Section 4, and we make a general discussion of the rates k_α for $\alpha > 0$ in Section 5. Section 6 discusses dissipative rate functions and prepares the way for the complete proof of the diffusive limit for $-1 \le \alpha \le 1$ in Section 7. The extensions are given in Section 8. An Appendix collects together some results about dissipativity, operators and the corresponding semigroups, whereas another one gives the technical proof of control of solutions in spaces with weights. This result is needed to control the decay of solutions at infinity.

3 Preliminaries

This section is devoted to recall some basic definitions as well as global existence and uniqueness results for the semilinear System (2), plus needed a priori estimates.

Let us begin by reviewing some notations. All data u_0, v_0 and solutions u, v in the paper are nonnegative. This means that $\rho \ge 0$, but j can have any sign. Equality of, or inequalities between integrable functions are understood in the almost everywhere sense (*a.e.*). The following usual notation will be adopted. For any function $f = f(x,t)$, we define its positive part f^+ by $f^+(x,t) = f(x,t)$ if $f(x,t) \ge 0$, and $f^+(x,t) = 0$ otherwise. We also define $f^-(x) = (-f)^+$, so that

$$f^+ = \frac{|f| + f}{2}, \quad f^- = \frac{|f| - f}{2}.$$

We define the functions $\mathrm{sign}\,(s) = 1$ if $s > 0$, $\mathrm{sign}\,(s) = -1$ if $s < 0$ and $\mathrm{sign}\,(s) = 0$ for $s = 0$; $\mathrm{sign}^+(s) = 1$ if $s > 0$, $\mathrm{sign}^+(s) = 0$ if $s \le 0$; and $\mathrm{sign}^-(s) = -1$ if $s < 0$, $\mathrm{sign}^-(s) = 0$ if $s \ge 0$. We have $\mathrm{sign}\,(s) = \mathrm{sign}^+(s) + \mathrm{sign}^-(s)$.

Moreover, for every smooth (or at least $W^{1,1}_{loc}$) function $f(x,t)$ we have a.e. $\partial|f|/\partial t = \mathrm{sign}\,(f)\,(\partial f/\partial t)$, as well as

$$\partial f^+(x,t)/\partial t = \mathrm{sign}^+(f)\,(\partial f/\partial t), \quad \partial f^-(x,t)/\partial t = \mathrm{sign}^-(f)\,(\partial f/\partial t).$$

We will often abbreviate for convenience a function $u(x,t)$ as u, or even $u(t)$ if only the t-dependence is meant.

We discuss next the interaction rates. In many of the arguments of the paper the interaction rate k need not be a power function, but we always assume that it satisfies some basic properties that we list next.

Definition 1 *We say that $k(u,v,x)$ is an* admissible interaction rate *if:*

R1. k is a measurable real function defined for $u, v \geq 0$ and $x \in \mathbb{R}$;

R2. for every $\lambda > 0$ there exists $M = M(\lambda) > 0$ such that $1/M \leq k(u,v,x) \leq M$ for all $1/\lambda \leq u, v \leq \lambda$, and $x \in \mathbb{R}$.

These minimal conditions are used in the literature, cf. e.g. [11]. They will be always assumed throughout the work and are satisfied by the typical rate functions $k = (u+v)^\alpha$ for all α. They have to be strengthened in some developments as follows.

Definition 2 *We say that k is a* regular interaction rate *if it is admissible and*

R3. $k(u,v,x)$ is continuous as a function of u and v for a.e. x. Moreover, the function $k(u,v,x)(v-u)$ is uniformly Lipschitz continuous as a function of u and v for bounded values of these arguments;

R4. for every $\lambda > 0$ there exist $M = M(\lambda)$ and $N = N(\lambda) > 0$ such that $N \leq k(u,v,x) \leq M$ for all $x \in \mathbb{R}$ and $0 \leq u, v \leq \lambda$;

It is easy to check that for the choice $k = k_\alpha = (u+v)^\alpha$, Conditions R1 and R2 hold, R3 is satisfied for $\alpha \geq 0$ and R4 is only satisfied for $\alpha = 0$. All are satisfied when k_α is replaced by $\bar{k}(u,v,x) = k_\alpha(u+\delta, v+\delta, x)$, $\delta > 0$.

Another property that plays a role is dissipativity.

Definition 3 *We say that an admissible rate k is* dissipative *(resp., ω-*dissipative*) if for every $a_1, a_2, b_2, b_2 \in \mathbb{R}$ and a.e. $x \in \mathbb{R}$*

$$(k_1\,(b_1 - a_1) - k_2\,(b_2 - a_2))\;(\,\text{sign}\,[a_1 - a_2] - \,\text{sign}\,[b_1 - b_2]) \leq 0,$$

with $k_1 = k(a_1, b_1, x)$, $k_2 = k_2(a_2, b_2, x)$; resp.

$$(k_1\,(b_1 - a_1) - k_2\,(b_2 - a_2))(\,\text{sign}\,[a_1 - a_2] - \,\text{sign}\,[b_1 - b_2]) \leq \omega(|a_1 - a_2| + |b_1 - b_2|).$$

This definition is motivated by the contractivity estimates, see Section 7, and has a functional framework that is discussed in Appendix I. It is shown there that, for power functions $k = (u+v)^\alpha$, k is dissipative if $|\alpha| \leq 1$. When $\alpha > 1$ it is only ω-dissipative for some ω that depends unfortunately on ε and does not allow to pass to the limit in the obtained estimates. If we replace sign by sign^+ in the formulas written above, we arrive at the concept of T-dissipativity, that is connected with comparison and will be surveyed later.

We address now the concept of weak solution, as well as existence and properties. We use the notation $Q_T = \mathbb{R} \times (0,T)$, for some $T > 0$. If $T = \infty$ we write $Q = \mathbb{R} \times (0,\infty)$.

Definition 4 *For given initial conditions $u_0(x)$, $v_0(x) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, we define a* weak solution *of System* (2) *as a pair of functions $(u,v) \in C([0,T] : L^p(\mathbb{R})) \cap L^\infty(Q_T)$, $T > 0$, for any $1 \leq p < \infty$, such that the equation is satisfied in the sense of distributions and the initial data are recovered in the sense of traces as $t \to 0$.*

We shall refer to weak solutions simply as solutions if no confusion is to be feared. Solutions are defined for $0 < t < T$. If $T = \infty$ we say that the solution is global (in time). Note that u and v are continuous functions of t with values in a function space.

As a starting point we state an existence and uniqueness result. Lions and Toscani [11] prove, in a slightly more general form, the following theorem:

Theorem 1 *Let $0 \le u_0(x), v_0(x) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and k be admissible. Then the initial value problem for System* (2) *admits a unique global weak solution*

$$u_\varepsilon(x,t), v_\varepsilon(x,t) \in L^\infty(Q) \cap C\left([0,\infty); L^p(\mathbb{R})\right).$$

These solutions enjoy a number of important properties; the following a-priori estimates are proved in [11]:

Lemma 1 *Let us suppose that $(u_\varepsilon(x,t), v_\varepsilon(x,t))$ is the solution of System* (2)*, with non-negative initial data $u_\varepsilon(x,0) = u_0(x)$, $v_\varepsilon(x,0) = v_0(x) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Then, for any regular convex function $\varphi(r)$, $r \ge 0$, we have*

$$\int_{\mathbb{R}} \left[\varphi(u_\varepsilon(x,t)) + \varphi(v_\varepsilon(x,t))\right] dx \le \int_{\mathbb{R}} \left[\varphi(u_0(x)) + \varphi(v_0(x))\right] dx. \tag{9}$$

In particular, by choosing $\varphi(r) = r$, we obtain the conservation of mass:

$$\int_{\mathbb{R}} \left[u_\varepsilon(x,t) + v_\varepsilon(x,t)\right] dx = \int_{\mathbb{R}} \left[u_0(x) + v_0(x)\right] dx. \tag{10}$$

Remark By taking $\varphi(r) = r^p$ for all $p \ge 1$, we obtain the boundedness of any L^p-norm

$$\int_{\mathbb{R}} \left[u(x,t)^p + v(x,t)^p\right] dx \le \int_{\mathbb{R}} \left[u_0(x)^p + v_0(x)^p\right] dx.$$

Moreover, when $p \to +\infty$, the lemma implies also the useful L^∞ bound:

$$u_\varepsilon(x,t), v_\varepsilon(x,t) \le \max\left\{\|u_0\|_\infty, \|v_0\|_\infty\right\}. \tag{11}$$

Complete order (Maximum Principle) will only be obtained later, under suitable conditions on the rate k.

For a particular choice of φ we also find an estimate of j_ε, which is called *entropy estimate*.

Lemma 2 *Let $(u_\varepsilon, v_\varepsilon)$ be the solution of System* (2)*, with initial data $u_0(x)$, $v_0(x) \in L^1 \cap L^\infty(\mathbb{R})$, $u_0(x), v_0(x) \ge 0$. Then there exists a constant $c_1 = c_1(u_0, v_0)$ such that*

$$\int_0^T \int_{\mathbb{R}} j_\varepsilon^2(x,t) \frac{k(u_\varepsilon, v_\varepsilon, x)}{(1+u_\varepsilon)(1+v_\varepsilon)}\, dxdt \le c_1. \tag{12}$$

Proof: We multiply the two equations of System (2) by $\varphi(u_\varepsilon)$ and $\varphi(v_\varepsilon)$ respectively, where $\varphi(r) = 1/(r+1)$, for $r \ge 0$. Adding the obtained equations and integrating on $\mathbb{R}$ we get

$$\frac{d}{dt} \int_{\mathbb{R}} \left[\log(u_\varepsilon + 1) + \log(v_\varepsilon + 1)\right] dx = \int_{\mathbb{R}} k(u_\varepsilon, v_\varepsilon, x) \frac{v_\varepsilon - u_\varepsilon}{\varepsilon^2} \frac{v_\varepsilon - u_\varepsilon}{(v_\varepsilon + 1)(u_\varepsilon + 1)} dx.$$

Since the second member of the previous equation is nonnegative, we prove that

$$\int_{\mathbb{R}} [\log(u_\varepsilon(x,t)+1) + \log(v_\varepsilon(x,t)+1)]\ dx$$

is a non-decreasing functional for System (2). By introducing the macroscopic quantities $\rho_\varepsilon = (u_\varepsilon + v_\varepsilon)$ and $j_\varepsilon = (u_\varepsilon - v_\varepsilon)/\varepsilon$, and using that $\log(r+1) \le r$, this proves that

$$0 \le \int_0^T \int_{\mathbb{R}} j_\varepsilon^2(x,t) \frac{k(u_\varepsilon, v_\varepsilon, x)}{(1+u_\varepsilon)(1+v_\varepsilon)}\, dxdt \le \int_{\mathbb{R}} [\log(u_\varepsilon(x,T)+1) + \log(v_\varepsilon(x,T)+1)]\ dx \le$$

$$\int_{\mathbb{R}} [u(x,T) + v(x,T)]\ dx = \int_{\mathbb{R}} [u_0(x) + v_0(x)]\ dx,$$

using in the last line the conservation of mass. □

The factor $k/(1+u_\varepsilon)(1+v_\varepsilon)$ can be eliminated from (12) under suitable assumptions to get a clean estimate of j^2. Indeed, if we take into account that u_ε and v_ε are of class $L^\infty(\mathbb{R})$ and are nonnegative, and if k is bounded below by a certain constant N then

$$\frac{k(u_\varepsilon, v_\varepsilon, x)}{(v_\varepsilon+1)(u_\varepsilon+1)} \ge \frac{N}{(\|\rho_0\|_\infty + 1)^2},$$

so that

$$\int_0^T \int_{\mathbb{R}} j^2(x,t)\, dxdt \le \frac{(\|\rho_0\|_\infty + 1)^2}{N} \int_{\mathbb{R}} [u_0(x) + v_0(x)]\ dx. \tag{13}$$

4 The hyperbolic system with locally bounded data

In this section we start our generalization of the conditions on the data by taking initial conditions $u_0(x)$, $v_0(x) \in L^\infty_{loc}(\mathbb{R})$ and constructing unique locally bounded solutions. Our analysis is based on the remark that the solutions of System (2) propagate along characteristics with speeds $\pm 1/\varepsilon$.

We will assume k to be an admissible rate function satisfying also the Lipschitz continuity condition R3. We have a dependence result in local L^∞ norm that justifies introducing the following generalized concept of solution.

Definition 5 *A weak solution of System* (2) *with locally bounded initial data* $u_0, v_0 \ge 0$ *is a pair of functions* $(u,v) \in C([0,T] : L^1_{loc}(\mathbb{R})) \cap L^\infty_{loc}(Q_T)$, $T > 0$, *such that the equation is satisfied in the sense of distributions and the initial data are recovered in the sense of traces.*

The following property holds:

Lemma 3 *Under the above assumptions on* k, *any weak solution is uniquely determined on an interval* $A = (a,b)$ *at a time* $t > 0$ *by the initial values taken on a larger interval*

$$A_0 = (a - (t/\varepsilon), b + (t/\varepsilon)). \tag{14}$$

Proof: It is an easy consequence of the finite propagation speed for a semilinear hyperbolic system. □

This result is very convenient because it means that we can construct solutions with locally bounded data, since the values that a weak solution takes in a shrinking region Γ depend only on the data in the initial interval $[a(0), b(0)]$.

Proposition 2 *Let* $0 \le u_0(x), v_0(x) \in L^\infty_{\text{loc}}(\mathbb{R})$ *and* k *be as above. Then the initial value problem for System* (2) *admits a unique solution* $u_\varepsilon(x,t)$, $v_\varepsilon(x,t) \in C\left([0,\infty); L^\infty_{\text{loc}}(\mathbb{R})\right)$.

Proof: The most natural construction is as follows. For every $n \in \mathbb{N}$ fixed, we take truncated initial conditions pair $(u_{0,n}(x), v_{0,n}(x))$ given by

$$u_{0,n}(x) = \begin{cases} u_0(x) & x \in [-n,n] \\ 0 & \text{otherwise} \end{cases} \qquad v_{0,n}(x) = \begin{cases} v_0(x) & x \in [-n,n] \\ 0 & \text{otherwise} \end{cases}$$

Thanks to Theorem 1, there exists a unique solution $(u_{\varepsilon,n}(x,t), v_{\varepsilon,n}(x,t))$ with those data. Given an interval $(-r,r)$ and a time $T > 0$ all the solutions with $n \ge r + (T/\varepsilon)$ coincide in the region $R = (-r,r) \times (0,T)$. Therefore, the limit

$$\lim_{n\to\infty} u_{\varepsilon,n}(x,t) = u_\varepsilon(x,t), \quad \lim_{n\to\infty} v_{\varepsilon,n}(x,t) = v_\varepsilon(x,t),$$

is well defined, locally bounded and a weak solution in the sense of our present definition. □

Remark Actually, we have proved that, from the point of view of local estimates, we can always assume that our initial data and solutions are compactly supported with respect to the space variable. This gives immediately local versions of the a priori estimates of Section 3, Lemmas 1 and 2. In Lemma 1 there is now the option that infinite mass is conserved. We ask the reader to check the easy details.

Uniform estimates

All these local estimates have a problem, namely that the region of dependence shrinks in time with speed $1/\varepsilon$, hence they are useless when passing to the limit $\varepsilon \to 0$. There is a way of obtaining ε-uniform estimates that we indicate next.

Lemma 4 *Let* u_ε *and* v_ε *be the solutions of System* (2)*, with initial data* $u_0(x)$, $v_0(x) \in L^\infty_{loc}(\mathbb{R})$, $u_0(x), v_0(x) \ge 0$*. Let* k *be admissible. Then for every test function* $\phi \in C_0^\infty(\mathbb{R})$, $\phi \ge 0$ *there exists a constant* $c_1 = c_1(u_0, v_0, \phi)$ *such that the solution of System* (2) *admits the following bound:*

$$\int_0^T \int_{\mathbb{R}} j_\varepsilon^2(x,t) k^2(u_\varepsilon, v_\varepsilon, x)\, \phi^2(x)\, dxdt \le c_1. \tag{15}$$

Proof: It is modification of the proof of Lemma 2. We multiply the two equations of System (2) by $\varphi(u_\varepsilon)$ and $\varphi(v_\varepsilon)$ respectively, where $\varphi(r) = 1/(r+1)$, for $r \ge 0$, and both by the square of $\phi(x)$. Adding the obtained equations and integrating we get

$$\frac{d}{dt}\int_{\mathbb{R}} [\log(u_\varepsilon + 1) + \log(v_\varepsilon + 1)]\, \phi^2(x)\, dx = \frac{2}{\varepsilon}\int_{\mathbb{R}} \log\left(\frac{u_\varepsilon + 1}{v_\varepsilon + 1}\right) \phi(x)\phi(x)_x\, dx +$$

$$\int_{\mathbb{R}} k(u_\varepsilon, v_\varepsilon, x) \frac{v_\varepsilon - u_\varepsilon}{\varepsilon^2} \frac{v_\varepsilon - u_\varepsilon}{(v_\varepsilon + 1)(u_\varepsilon + 1)} \phi^2(x)\, dx.$$

Since u, $v \geq 0$, then for all $t > 0$

$$\int_{\mathbb{R}} [\log(u_\varepsilon(x,t) + 1) + \log(v_\varepsilon(x,t) + 1)]\, \phi(x)^2\, dx \geq 0.$$

We also have

$$k(u_\varepsilon, v_\varepsilon, x) \frac{v_\varepsilon - u_\varepsilon}{\varepsilon^2} \frac{v_\varepsilon - u_\varepsilon}{(v_\varepsilon + 1)(u_\varepsilon + 1)} \phi^2 \geq \frac{j_\varepsilon^2 k_\varepsilon^2}{M(v_\varepsilon + 1)(u_\varepsilon + 1)} \phi^2,$$

where M is the supremum of k_ε. The new term in ϕ_x is treated as follows. Assume that at a certain point $u_\varepsilon \geq v_\varepsilon$. Then

$$0 \leq \frac{1}{\varepsilon} \log\left(\frac{u_\varepsilon + 1}{v_\varepsilon + 1}\right) \leq \frac{j_\varepsilon}{1 + v_\varepsilon}.$$

For any $\gamma > 0$ we can now write

$$\frac{2}{\varepsilon} \log\left(\frac{u_\varepsilon + 1}{v_\varepsilon + 1}\right) \phi\phi_x \leq \gamma \frac{j_\varepsilon^2 k_\varepsilon^2 \phi^2}{N^2(1 + v_\varepsilon)(1 + u_\varepsilon)} + \frac{1}{\gamma} \frac{(1 + u_\varepsilon)}{(1 + v_\varepsilon)} \phi_x^2,$$

where $k_\varepsilon = k(u_\varepsilon, v_\varepsilon, x)$ and N is the infimum of k_ε. Similarly when $u_\varepsilon \leq v_\varepsilon$. Putting all together with $\gamma = N^2/(2M)$ and integrating in t we get

$$\int\int_{Q_T} \frac{j_\varepsilon^2 k_\varepsilon^2 \phi^2}{(1 + v_\varepsilon)(1 + u_\varepsilon)}\, dxdt \leq C_1 \int\int_{Q_T} \phi_x^2\, dxdt +$$

$$C_2 \int_{\mathbb{R}} [\log(u_\varepsilon(x,T) + 1) + \log(v_\varepsilon(x,T) + 1)]\, \phi^2\, dx,$$

where $C_2 = 2M$ and $C_1 = 4M^2(1 + \|\rho_0\|_\infty)/N^2$. Since u_ε and v_ε are bounded the conclusion follows. □

Note that when k is bounded below away from zero we obtain a local estimate for j_ε in $L^2_{x,t}$. This remark will be important in the next section.

5 Limiting behavior with a regular interaction rate

After these considerations we will solve System (2) $u_\varepsilon(x,0) = u_0(x)$ and $v_\varepsilon(x,0) = v_0(x)$ in the class of bounded solutions and initial data if the system has a regular interaction rate in the sense of Definition 2, Section 3. This implies that there are suitable a priori estimates on the solutions u_ε, v_ε (or, equivalently, ρ_ε and j_ε), needed to perform the passage to the limit $\varepsilon \to 0$.

Let us write $K(\rho, x) = k(\rho/2, \rho/2, x)$. We denote by $S = S_{R,T}$ the bounded strips $(-R, R) \times (0, T)$ in space time. We will prove the following result.

Theorem 3 *Let $(u_\varepsilon, v_\varepsilon)$ a sequence of solutions for the initial value problem of System (2) with initial values $u_\varepsilon(x,0) = u_{0\varepsilon}(x)$, $v_\varepsilon(x,0) = v_{0\varepsilon}(x)$, $u_{0\varepsilon}(x), v_{0\varepsilon}(x) \in L^\infty(\mathbb{R})$ such that $0 \leq u_{0\varepsilon}(x), v_{0\varepsilon}(x) \leq M$ and*

$$u_{0\varepsilon} \rightharpoonup u_0, \quad v_{0\varepsilon} \rightharpoonup v_0$$

in the sense of weak convergence in $L^1_{loc}(\mathbb{R})$*. If* k *is a regular interaction rate, then there exists a function*

$$\rho(x,t) \in L^\infty(Q) \cap C(Q), \quad \rho(x,t) \geq 0,$$

such that $\rho_\varepsilon = u_\varepsilon + v_\varepsilon$ *converges strongly to* $\rho(x,t)$ *in* $L^2(S)$*, for every bounded strip* S*, and the limit density* $\rho(x,t)$ *is the unique bounded weak solution of the Cauchy problem for the nonlinear diffusion equation*

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x}\left(D(\rho)\frac{\partial \rho}{\partial x}\right). \tag{16}$$

with diffusivity $D(\rho) = 1/(2K(\rho))$*, and taking the initial data* $\rho_0(x) = u_0(x) + v_0(x)$ *in the sense of traces. Moreover,* j_ε *is uniformly bounded in* $L^2(S)$*. It follows that* $u_\varepsilon(x,t) \to \rho(x,t)/2$*,* $v_\varepsilon(x,t) \to \rho(x,t)/2$*, and* $\varepsilon j_\varepsilon(x,t) \to 0$ *a.e. and strongly in* $L^2(S)$ *for every* S*.*

A remark about convergence. Since the data are uniformly bounded, weak convergence in $L^1_{loc}(S)$ is equivalent to weak convergence in $L^p_{loc}(S)$ for any $p > 1$, or to weak-star convergence in $L^\infty(S)$. The solutions are also uniformly bounded, so that convergence in $L^2_{loc}(Q)$ implies immediately convergence of ρ_ε in $L^p_{loc}(\mathbb{R})$ for all $p < \infty$. Moreover, if the initial data are continuous, the theory of nonlinear diffusion equations says that the data ρ_0 are taken in plain continuous sense.

Proof: Let us collect the information we have on the family of solutions of the present good problems for different $\varepsilon > 0$.

By the uniform boundedness of the solutions, ρ_ε is bounded in $L^2(S)$, and therefore there is a subsequence that converges weakly to a limit ρ in all strips S.

Under our assumptions we may state the entropy estimate in a local form as follows:

Lemma 5 *Let* u_ε *and* v_ε *be the solutions of System* (2)*, with initial data* $u_0(x)$*,* $v_0(x) \in L^\infty_{loc}(S_{R,T})$*. Let* k *be a regular interaction rate and* L *be the Lipschitz constant of* $f(u,v,x)$ *prescribed by Condition R3 of Definition 2. Then for every* $R > 0$ *there exists a constant* $c = c(u_0, v_0, L, N) > 0$*, independent of* ε*, such that the solution of System* (2) *admits the following bound:*

$$\int_0^T \int_{-R}^R j_\varepsilon(x,t)^2 \, dxdt \leq c. \tag{17}$$

We deduce that j_ε converges, at least along subsequences, to some j weakly in $L^2_{x,t}(S)$ for all S.

Moreover, $k(u_\varepsilon, v_\varepsilon, x) j_\varepsilon$ is also locally bounded in $L^2_{x,t}$ and converges weakly locally to some limit w.

Div-Curl. In order to obtain the precise characterization of this limit we need to use the following basic tool of compensated compactness theory known as the *div-curl lemma* [14]:

Lemma 6 *Let* A *be an open set in* $\mathbb{R}^n$*, and let* p_ε*,* q_ε *two sequences such that*

1. $p_\varepsilon \rightharpoonup p$ *in* $\left[L^2(A)\right]^n$*;*
2. $q_\varepsilon \rightharpoonup q$ *in* $\left[L^2(A)\right]^n$*;*

3. $\operatorname{div} p_\varepsilon$ *is bounded in* $L^2(A)$*;*

4. $\operatorname{curl} q_\varepsilon$ *is bounded in* $\left[L^2(A)\right]^n$*.*

Then, if we indicate with $\langle \cdot, \cdot \rangle$ *the scalar product in* $\mathbb{R}^n$*, that is* $\langle p, q \rangle = \sum_{i=1}^n p_i q_i$*, then*

$$\langle p_\varepsilon, q_\varepsilon \rangle \to \langle p, q \rangle \text{ in } \mathcal{D}'.$$

We can now apply the div-curl lemma: let $p_\varepsilon = (\rho_\varepsilon, j_\varepsilon)$ and $q_\varepsilon = (-\rho_\varepsilon, \varepsilon^2 j_\varepsilon)$ taking as A any bounded strip in space time of the form $S = (-R, R) \times (0, T)$. The first equation of the our hyperbolic system means exactly that $\operatorname{div} p_\varepsilon = 0$, whereas the second one gives that $\operatorname{curl} q_\varepsilon$ is bounded in L^2, because of the boundedness of $k(\rho_\varepsilon) j_\varepsilon$. Therefore,

$$\langle p_\varepsilon, q_\varepsilon \rangle = -\rho_\varepsilon^2 + \varepsilon^2 j_\varepsilon^2 \;\text{ converges to }\; \langle p, q \rangle = -\rho^2$$

in $\mathcal{D}'$ as $\varepsilon \to 0^+$. Since ρ_ε^2 is bounded in L^2 we can then deduce, by the uniqueness of the distributional limit, that ρ^2 belongs to $L^2_{x,t}$.

We also know that $\rho_\varepsilon \rightharpoonup \rho$ in $L^2_{x,t}$. We then conclude that ρ_ε converges strongly thanks to the following known result:

Lemma 7 *Let* Ξ *be a set of finite measure and* ρ_ε *a sequence of functions of class* $L^2(\Xi)$ *such that* $\rho_\varepsilon \rightharpoonup \rho$ *in* $L^2(\Xi)$ *and* $\rho_\varepsilon^2 \rightharpoonup \rho^2$ *in* $L^2(\Xi)$*. Then* $\rho_\varepsilon \to \rho$ *in* $L^2(\Xi)$*.*

From the previous result, we finally deduce the following convergences: $\rho_\varepsilon \to \rho$, $u_\varepsilon \to \rho/2$ and $v_\varepsilon \to \rho/2$ in $L^2_{loc}(Q)$ along a suitable subsequence $\varepsilon \to 0$; moreover, $k(u_\varepsilon, v_\varepsilon, x) j_\varepsilon \to K(\rho, x) j$ in $L^1_{loc}(Q)$.

Indeed, the convergence of u_ε and v_ε in L^2_{loc} comes from the convergence of ρ_ε and the fact that $\varepsilon j_\varepsilon \to 0$ in L^2_{loc}. This implies that for every strip S and along a subsequence u_ε and v_ε converge a.e. to the common limit $\rho/2$. Using the boundedness we conclude that the convergence of $k(u_\varepsilon, v_\varepsilon, x)$ towards $k(\rho/2, \rho/2, x)$ takes place a.e and in $L^2(S)$ strong. Since j_ε converges weakly in L^2_{loc}, the last assertion follows.

Passage to the limit. The convergence of both ρ_ε and j_ε implies that we may pass to the limit in the first equation of System (2), which is linear, and

$$\frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} = 0$$

in the sense of distributions. From the second equation of (2) we deduce that

$$\frac{\partial \rho}{\partial x} = -2K(\rho) j, \tag{18}$$

at least in distributional sense. Indeed, for any test-function $\phi \in \mathcal{D}(\mathbb{R} \times (0, T))$ we have:

$$\varepsilon^2 \left[\int \int j_\varepsilon \frac{\partial \phi}{\partial t}\, dx dt + \int j_\varepsilon(x, 0) \phi(x, 0)\, dx \right] +$$
$$+ \int \int \rho_\varepsilon \frac{\partial \phi}{\partial x}\, dx dt = 2 \int \int k(u_\varepsilon, v_\varepsilon, x) j_\varepsilon \phi\, dx dt.$$

Since $\varepsilon^2 j_\varepsilon \to 0$ in L^2_{loc}, as well as $\varepsilon^2 \int j_\varepsilon(x,0)\phi(x,0)dx$, we obtain in the limit:

$$\int\int \frac{\partial \rho}{\partial x}\phi \, dxdt = -2\int\int K(\rho) j \phi \, dxdt,$$

which is the weak formulation of (18) in $\mathcal{D}'$. This also means that $\partial\rho/\partial x$ is an L^2_{loc} function. Now, since k is bounded above and below away from zero, we deduce from (18) that

$$j = -\frac{1}{2K(\rho)}\frac{\partial \rho}{\partial x} \tag{19}$$

in $L^2_{loc}(Q)$. By inserting equation (19) in the continuity equation, we can finally conclude that ρ satisfies the nonlinear diffusion equation (16) with diffusivity $D(\rho) = 1/2K(\rho)$.

Weak equation with initial data. Let $\rho_{0\varepsilon}(x) = u_{0\varepsilon}(x) + v_{0\varepsilon}(x)$. By the first equation of the system we have

$$\int_0^T \int_{\mathbb{R}} (\rho_\varepsilon \eta_t + j_\varepsilon \eta_x) \, dxdt + \int_{\mathbb{R}} \rho_{0\varepsilon}(x)\eta(x,0)\, dx = 0$$

for all test functions $\eta(x,t) \in C^\infty_{x,t}(Q)$ such that $\eta \geq 0$ and η vanishes for all $|x| \geq R$ and $t \geq T$. Using the convergences of ρ_ε and j_ε as $\varepsilon \to 0$ and the expression for j we get the complete form of the weak formulation

$$\int_0^T \int_{\mathbb{R}} \left(\rho\eta_t - \frac{1}{K(\rho)}\rho_x \eta_x\right) \, dxdt + \int_{\mathbb{R}} \rho_0(x)\eta(x,0)\, dx = 0 \tag{20}$$

for the same class of functions η. This is a standard way of incorporating the initial data into the weak formulation in weak theories of parabolic equations.

Identifying the limit. Equation (16), with initial conditions of the type described by (20), is uniformly parabolic, and therefore admits one and only one global solution which is a continuous function for $t > 0$. See [10] for the classical quasilinear theory and [15] for uniqueness for nonlinear diffusion equations. This uniqueness result guarantees the existence of a unique limit point for the whole family $\{\rho_\varepsilon\}$. The regularity theory for quasilinear parabolic equations implies that ρ is Hölder continuous for $t > 0$. It is also continuous down to $t = 0$ if the initial data $\rho \in C(\mathbb{R})$.

Initial trace. Next, we prove that the limit equation takes on the initial condition when $t \to 0$ in the weak sense (trace sense). By the first equation of the system for every $\varphi \in C^1_0(\mathbb{R})$ and for every $\tau > 0$ we have

$$\int_0^\tau \int_{\mathbb{R}} \left(\frac{\partial \rho_\varepsilon}{\partial t} + \frac{\partial j_\varepsilon}{\partial x}\right)\varphi(x)\, dxdt = 0.$$

Hence, putting $\rho_{0\varepsilon}(x) = u_{0\varepsilon}(x) + v_{0\varepsilon}(x)$ we have

$$\int_{\mathbb{R}} (\rho_\varepsilon(x,\tau) - \rho_{0\varepsilon}(x))\,\varphi(x)\, dx = \int_0^\tau \int_{\mathbb{R}} j_\varepsilon \frac{\partial \varphi}{\partial x}\, dxdt. \tag{21}$$

Let us show that the right-hand side vanishes as $(\varepsilon,\tau) \to (0^+,0^+)$. Indeed,

$$\int_0^\tau \int_{\mathbb{R}} j_\varepsilon \frac{\partial \varphi}{\partial x}\, dxdt \leq \left(\int_0^\tau \int_{\mathbb{R}} j^2_\varepsilon(x,t)\, dxdt\right)^{1/2} \left(\int_0^\tau \int_{\mathbb{R}} \left|\frac{\partial \varphi}{\partial x}\right|^2 \, dxdt\right)^{1/2}$$

$$\le C\tau^{1/2}\left(\int_0^\tau\int_{\mathbb{R}} j_\varepsilon^2(x,t)\,dxdt\right)^{1/2} \le C^*\,\tau^{1/2}.$$

Therefore, we can pass to the limit $\varepsilon \to 0^+$ in (21) and get for almost all $t > 0$

$$\int_{\mathbb{R}} (\rho(x,t) - \rho_0(x))\,\varphi(x)\,dx \le C^*\,t^{1/2}.$$

This holds for all t by continuity of ρ. Letting $t \to 0$ we obtain

$$\lim_{t\to 0^+}\int_{\mathbb{R}} (\rho(x,t) - \rho_0(x))\,\varphi(x)\,dx = 0, \tag{22}$$

for all $\varphi \in C_0^1(\mathbb{R})$. Since the functions $\rho(x,t)$, $\rho_0(x)$ are uniformly bounded, we can extend this limit property to test functions $\varphi(x) \in L^1(\mathbb{R})$ by approximation. In other words, the initial data are recovered in the sense of weak $L^1_{loc}(\mathbb{R})$ convergence. □

Better convergence can be obtained if the rate or the data have additional properties. We will examine both issues in the sequel. To begin with, we may get better convergence by a control of the tails of the solutions $|x| \to \infty$.

Lemma 8 *Let u_0, $v_0 \in L^1_+(\mathbb{R}) \cap L^\infty(\mathbb{R})$, such that the moments of order 2β exist and are finite:*

$$\int_{\mathbb{R}} (u_0(x) + v_0(x))(1+|x|^2)^\beta\,dx < C_0 \tag{23}$$

for some $0 < \beta < 1/4$. Then for every $0 < t < T$ we have that the pair $(u_\varepsilon, v_\varepsilon)$, solution of (2), satisfies the estimate

$$\int_{\mathbb{R}} (u_\varepsilon(x,t) + v_\varepsilon(x,t))(1+|x|^2)^\beta\,dx < C_T. \tag{24}$$

Proof: We calculate

$$\frac{d}{dt}\int_{\mathbb{R}} \rho_\varepsilon(x,t)(1+|x|^2)^\beta\,dx = 2\beta\int_{\mathbb{R}} j_\varepsilon\,x(1+|x|^2)^{\beta-1}\,dx.$$

Since the data are in L^1 we know that $j_\varepsilon \in L^2(Q_T)$. On the other hand, $x(1+|x|^2)^{\beta-1} \in L^2(Q_T)$ if $\beta < 1/4$. □

Generally speaking, the result cannot be applied to the system with rate $k_\alpha = (u+v)^\alpha$ with $\alpha \neq 0$. However, if we know that the data and the solutions are also bounded away from zero, then such rate k_α fulfills all needed conditions on the range of applied values for every $\alpha \in \mathbb{R}$ and we have:

Corollary 4 *Theorem 3 applies to the systems with rate k_α for all α under the additional assumption: $u_\varepsilon, v_\varepsilon \ge \delta > 0$ for every ε.*

Having a finite moment implies integrability, and moreover, uniform integrability at infinity. We notice that under the joint conditions of Theorem 3 and Lemma 8 the constant C_T is uniform in ε, hence the sequence ρ_ε has a uniformly small integral in exterior sets of the form $\{|x| \ge R\} \times (0,T)$ for R large enough. This allows to improve the results of Theorem 3.

Corollary 5 *Under the conditions of Theorem* 3*, if the initial data are integrable and the moments of order* 2β *given by* (23) *with* $0 < \beta < 1/4$ *are uniformly bounded, then* $\rho_\varepsilon \to \rho$*,* $u_\varepsilon \to u$ *and* $v_\varepsilon \to v$ *strongly in* $L^1(Q_T)$*.*

We will discuss merely integrable data after we introduce the question of dissipativity in the next section. More results on tail control will be needed in Section 8 and are given in Appendix 2. Finally, in the next section we will need an improvement of Theorem 3:

Corollary 6 *The result of Theorem* 3 *still holds if we allow to replace the rate* k *in the system satisfied by* $u_\varepsilon, v_\varepsilon$ *by a uniformly bounded regular family* k_ε *such that* $k_\varepsilon \to k$ *a.e..*

By a uniformly bounded family we mean that the constants in the definition of regular rate are the same for all ε.

6 Hydrodynamical limit for rates k_α, $\alpha > 0$. General results

Let us examine next the hydrodynamical limit for nonregular rates. We take as study case the rates k_α with $\alpha > 0$ that are degenerate at the level $u = 0$ so that we cannot use the method of the previous section for small data, like $u_0, v_0 \in X = L^1_+(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

As we had proposed, we lift the initial data by an amount $\delta > 0$, solve the approximate problems to get solutions $\rho_{\varepsilon,\delta}(x,t), j_{\varepsilon,\delta}(x,t)$, corresponding to shifted data $(u_{0\varepsilon}(x)+\delta, v_{0\varepsilon}(x)+\delta)$. The next step is to pass to the limit $\varepsilon \to 0^+$ and $\delta \to 0^+$ to recover the hydrodynamical limit for

$$\rho_\varepsilon(x,t) = \lim_{\delta\to 0^+} \rho_{\varepsilon,\delta}(x,t).$$

This is a delicate case of double limit, and we will not be able to solve it in its full generality unless $\alpha \le 1$, thanks to the properties of ordered dissipativity.

In any case, one of the iterated limits is easy to study. Theorem 3 solves the diffusive limit of the lifted problems. As $\varepsilon \to 0^+$, $\rho_{\varepsilon,\delta} \to \rho_\delta$ for fixed $\delta > 0$, where ρ_δ solves the diffusion equation with data $\rho_{0\delta}(x) = \rho_0(x) + \delta$.

The next step is to pass to the limit as $\delta \to 0^+$. This needs to know the properties of diffusion equations that we recall next. Indeed, weak solutions can be constructed for the diffusion equation with quite general data, say, L^∞_{loc} data. The Maximum Principle applies to the diffusion equation, so that the family ρ_δ is monotone, and there exists the limit

$$\rho(x,t) = \lim_{\delta\to 0} \rho_\delta(x,t) \ge 0.$$

The value of $\rho(x,t)$ has been carefully studied in the fast diffusion literature and the situation is as follows:

(a) *Standard diffusive case:* When either the exponent $\alpha < 2$ or the data ρ_0 are not integrable (both as $x \to \infty$ and as $x \to -\infty$). In that case it is known that the limit $\rho(x,t)$ is the solution of the diffusion equation with initial data $\rho_0(x)$. For data in the class X the limit takes place in $L^p_{loc}(\mathbb{R})$ for every $p \in [1,\infty)$, uniformly in time. Note that standard norm used in nonlinear diffusion theory is the L^1-norm, but our solutions are uniformly bounded, hence, all L^p norms apply as well.

(b) *Ultra-diffusive case:* it happens when $\alpha \ge 2$ and the data are integrable at one of the two ends of the real line. Then the limit $\rho(x,t) = 0$ for every $x \in \mathbb{R}$ and $t > 0$, in other words, it is the trivial solution. In physical terms, this interesting limit means that the mass moves out to infinity at increasing speeds as $\delta \to 0$, so that the "limit process is empty because all the mass went to infinity" at $t = 0^+$. The discontinuity in the initial data is called an initial layer. Convergence is locally uniform in $\mathbb{R} \times [\tau, \infty)$ for $\tau > 0$.

Proposition 7 *Given $\alpha > 0$ and $u_0, v_0 \in L^\infty(\mathbb{R})$, the iterated limit*

$$\rho(x,t) = \lim_{\delta\to 0}\lim_{\varepsilon\to 0} \rho_{\varepsilon,\delta}(x,t) \tag{25}$$

of the lifted approximations to the hyperbolic system (2) *is the solution of the diffusion equation in case* (a)*, identically zero in case* (b)*. The convergence takes place in $L^2_{loc}(\mathbb{R} \times [\tau, \infty))$ with $\tau = 0$ in case* (a)*, $\tau > 0$ in case* (b)*.*

Better convergence can be obtained by looking more closely to the data but this is not our main concern here. It is rather to improve the result if possible to a complete double limit $(\varepsilon, \delta) \to (0, 0)$, or at least to a wider classes of sequences. The whole limit will only be achieved when $|\alpha| \le 1$. But the existence of special sequences along which both ε and δ go to zero can be done in the present general context.

Proposition 8 *There exists a monotone continuous function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{s\to 0} \omega(s) = 0$ such that the limit of Theorem 7 holds along all sequences $(\varepsilon, \delta) \to (0, 0)$ with $\varepsilon, \delta > 0$, with the restriction that $\varepsilon \le \omega(\delta)$.*

Proof: Let us fix a set $S = (-r, r) \times [\tau, T]$ with $r, \tau, T > 0$, and $\tau = 0$ in case (a), $0 < \tau < T$ in case (b). Given any $n \ge 1$ and we first find a $\delta_n > 0$ such that

$$\int_{-r}^{r} |\rho_{\delta_n}(x,t) - \rho(x,t)|\, dx \le 1/n,$$

for all $\tau < t \le T$. We can choose the δ_n as a decreasing sequence. We define the intervals $J_n = [\delta_{n+1}, \delta_n]$.

As a second step, let us consider the functions $\bar\rho_{\varepsilon,\delta}(x,t) = \rho_{\varepsilon,\delta} - 2\delta$ with $\delta \in J_n$, n fixed. Every $\bar\rho_{\varepsilon,\delta}$ solves system (2) with $k = (u_\varepsilon + v_\varepsilon + 2\delta)^\alpha$, which is regular. Moreover, the constants that are used in the proof of Theorem 3 are uniform for $\delta \in J_n$, hence the convergence result of this theorem can be extended to cover also the case where k varies in the present way, see Corollary 6. This is a simple but important verification that we leave to the reader, which will also observe that ε_n depends on n, r, J_n, and the maximum M of ρ but not on the particular data.

The consequence of the previous step is that there exists ε_n such that

$$\|\rho_{\varepsilon_n,\delta} - \rho_\delta\|_2 \le \frac{1}{n}$$

in the L^2 norm in space-time on S. Without loss of generality we may choose ε_n as a decreasing sequence as $n \to \infty$.

We now define $\omega(\delta)$ in any continuous way so that $\omega(\delta) \le \varepsilon_n$ whenever $\delta \in J_n$. □

Remark There is a weak point in this result: namely, we have only a positive result along special sequences, and the condition $0 < \varepsilon \le \omega(\delta)$ defining them is not explicit.

7 Dissipative rate functions. L^1 estimates, mild solutions and Maximum Principle

In the next sections we examine the improvements that may be obtained by assuming that the rate is dissipative, see Definition 3. The main fact is that we can obtain dependence results expressed in terms of L^1 norms instead of L^∞ norms. Our first result is a version of Lemma 3.

Lemma 9 *Let k be admissible and ω-dissipative and let $(u_i(x,t), v_i(x,t))$ be two weak solutions of (2) with initial data $(u_{0,i}(x), v_{0,i}(x))$, $i = 1, 2$. Then, for every $a, b \in \mathbb{R}$, $a < b$, we have*

$$\begin{aligned} &\int_a^b (|u_1(x,t) - u_2(x,t)| + |v_1(x,t) - v_2(x,t)|)\, dx \le \\ &e^{\omega t/\varepsilon^2} \int_{a-t/\varepsilon}^{b+t/\varepsilon} (|u_{0,1}(x) - u_{0,2}(x)| + |v_{0,1}(x) - v_{0,2}(x)|)\, dx. \end{aligned} \tag{26}$$

Proof: The outline is the same of Lemma 3. At the end of the proof the contribution of the right-hand terms involving k disappears precisely because of dissipativity. If k is only ω-dissipative the end of proof is very similar and we have to use Gronwall's Lemma with constant ω/ε^2. $\square$

When the data are integrable we can let $a, b \to \infty$. We have

Corollary 9 *If k is dissipative and the data are integrable then for every $t > 0$ and any two weak solutions of (2) we have*

$$\|u_1(t) - u_2(t)\|_1 + \|v_1(t) - v_2(t)|_1 \le e^{\omega t/\varepsilon^2}(\|u_{0,1} - u_{0,2}\|_1 + \|v_{0,1} - v_{0,2}\|_1), \tag{27}$$

where $\|\cdot\|_1$ denotes the norm in $L^1(\mathbb{R})$.

When $\omega = 0$ real contraction (better said, non-expansion) happens in $L^1(\mathbb{R})$ norm. This holds in our problem when $k(u, v, x) = (u + v)^\alpha$ with $|\alpha| \le 1$.

Mild solutions with integrable data

Using the latter result, we may go back to the hyperbolic system and extend Theorem 1 to cover all initial data in the class $L^1_+(\mathbb{R})$ without the restriction of local boundedness. But for that we also have to extend the class of solutions.

Definition 6 *A mild solution of System (2) is a pair of functions $(u, v) \in C([0, T] : L^1(\mathbb{R}))$, that can be obtained as limit in $C([0, T] : L^1(\mathbb{R}))$ of a sequence (u_n, v_n) of weak solutions of the system with data $(u_{0n}, v_{0n}) \in L^1_+(\mathbb{R}) \cap L^\infty(\mathbb{R})$.*

Here, T is a finite number or $+\infty$ depending on the problems statement. The distinction is important for problems which may have blow-up in finite time, which is not the case in our study. As a consequence of the previous results, if k is ω-dissipative such a limit exists for every convergent sequence of data and is uniquely determined by the limit of the initial data. We have (cf. e.g. [4] or [8] for the theory of dissipative operators)

Theorem 10 *Let $0 \le u_0(x), v_0(x) \in L^1(\mathbb{R})$ and k be ω-dissipative. Then the initial value problem for System (2) admits a unique mild solution $u_\varepsilon(x,t), v_\varepsilon(x,t) \in C\left([0,\infty); L^1(\mathbb{R})\right)$ for all $T > 0$. Estimate (27) holds for any two mild solutions.*

T-dissipativity and comparison

The rate functions $k(u,v,x) = (u+v)^\alpha$ with $|\alpha| \leq 1$ enjoy a stronger version dissipativity, called T-dissipativity.

Definition 7 *We say that an admissible rate k is T-dissipative (resp. T-ω-dissipative) if for every $a_1, a_2, b_2, b_2 \in \mathbb{R}$ and a.e. $x \in \mathbb{R}$ we have*

$$(k(a_1,b_1,x)(b_1-a_1) - k(a_2,b_2,x)(b_2-a_2))(\operatorname{sign}^+[a_1-a_2] - \operatorname{sign}^+[b_1-b_2]) \leq 0,$$

resp.

$$(k(a_1,b_1,x)(b_1-a_1) - k(a_2,b_2,x)(b_2-a_2))(\operatorname{sign}^+[a_1-a_2] - \operatorname{sign}^+[b_1-b_2]) \leq$$
$$\omega(|a_1-a_2|^+ + |b_1-b_2|^+).$$

We can prove a variant of Lemma 9 adapted to this concept.

Lemma 10 *Let k be admissible and T-ω-dissipative and let $(u_i(x,t), v_i(x,t))$ be two weak solutions of (2) with initial data $(u_{0,i}(x), v_{0,i}(x))$, $i = 1, 2$. Then, for every $a, b \in \mathbb{R}$, $a < b$, we have*

$$\begin{aligned}
&\int_a^b \left(|u_1(x,t) - u_2(x,t)|^+ + |v_1(x,t) - v_2(x,t)|^+\right) dx \leq \\
&e^{\omega t/\varepsilon^2} \int_{a-t/\varepsilon}^{b+t/\varepsilon} \left(|u_{0,1}(x) - u_{0,2}(x)|^+ + |v_{0,1}(x) - v_{0,2}(x)|^+\right) dx.
\end{aligned} \tag{28}$$

Proof: We argue as in Lemma 9, by using $\operatorname{sign}^+[u_1(x,t) - u_2(x,t)]$ instead of $\operatorname{sign}[u_1(x,t) - u_2(x,t)]$ and $\operatorname{sign}^+[v_1(x,t) - v_2(x,t)]$ instead of $\operatorname{sign}[v_1(x,t) - v_2(x,t)]$ at every occurrence of sign function in the proof. □

The importance of this result lies, in particular, in the following immediate consequence.

Corollary 11 *Under the previous assumptions, if the data are ordered: $u_{1,0}(x) \leq u_{2,0}(x)$ and $v_{1,0}(x) \leq v_{2,0}(x)$ almost everywhere, then the solutions of (2) are also ordered:*

$$u_1(x,t) \leq u_2(x,t), \qquad v_1(x,t) \leq v_2(x,t) \tag{29}$$

almost everywhere in x, for all $t \in \mathbb{R}^+$.

This is the Comparison Theorem, a powerful tool in Differential Equations usually referred to as the *Maximum Principle.* We recall that $k(u,v,x) = (u+v)^\alpha$ is T-dissipative for $|\alpha| \leq 1$, see Appendix I.

8 Initial data in the class L^1_+ for k_α, $|\alpha| \le 1$

Next, we want to consider the case where the rate is T-dissipative, and its δ-lifting regular, for instance k_α with $|\alpha| \le 1$. In order to avoid technicalities we consider only the case k_α. We can treat general nonnegative and integrable data. We are now able to complete the proof of the double limit addressed in Section 6. We can even eliminate other requirements, like boundedness of the solutions.

We start our study with the L^1 theory. This is our main result.

Theorem 12 *Let $k = k_\alpha$ with $|\alpha| \le 1$, and let $(u_\varepsilon, v_\varepsilon)$ a sequence of mild solutions for the initial value problem of System* (2)*, with nonnegative initial values $u_{0\varepsilon}(x,0) \in L^1(\mathbb{R})$, $v_{0\varepsilon}(x,0) \in L^1(\mathbb{R})$ converging to some u_0, $v_0 \in L^1(\mathbb{R})$ as $\varepsilon \to 0$. The following holds:*

(a) There exists a positive and smooth function ρ such that $\rho_\varepsilon(x,t)$ converges to $\rho(x,t)$ in $C([0,T]; L^1(\mathbb{R}))$, $T \ge 0$.

(b) When $-1 \le \alpha < 1$, the limit density $\rho(x,t)$ is the unique weak solution of the Cauchy problem for the nonlinear heat equation (5) *with initial data $\rho_0(x) = u_0(x) + v_0(x) \in L^1(\mathbb{R})$. If $\alpha = 1$, we obtain the unique maximal solution of* (5)*, characterized by the property of conservation of mass.*

(c) We also have $u_\varepsilon, v_\varepsilon \to \rho/2$ in $L^p(S)$ on compact subsets S of $Q = \mathbb{R} \times (0,\infty)$, $p < \infty$.

Proof: **I.** *Case of special data.* We make the additional assumptions that u_0, $v_0 \in L^1_+(\mathbb{R}) \cap L^\infty(\mathbb{R})$, and that there is sufficient decay as $|x| \to \infty$, for instance:

$$\int_{\mathbb{R}} (1+x^2)^p (u_{0\varepsilon}^2(x) + v_{0\varepsilon}^2(x))\, dx < +\infty \qquad (30)$$

for some $p > 1/2$, uniformly in $\varepsilon > 0$. We proceed in three steps:

(i) As in Section 6, we approximate solutions by lifting the data of the problem by an amount $\delta > 0$, getting solutions $\rho_{\varepsilon,\delta}(x,t) \ge 2\delta$, bounded above by $\|\rho\|_\infty + 2\delta$. By the law of mass conservation (10) we have

$$\int_{\mathbb{R}} (\rho_{\varepsilon,\delta}(x,t) - 2\delta)\, dx = \int_{\mathbb{R}} \rho_{0\varepsilon}(x) dx.$$

In order to see the limit of the original problem we review the double limit with respect to ε and δ, beginning with the iterated limit already discussed.

First, we let $\varepsilon \to 0^+$ with $\delta > 0$ fixed, obtaining ρ_δ in the process. We then have to let $\delta \to 0^+$. This involves only the target equation (5). Completing what was said in Section 6, there is uniqueness and comparison for uniformly positive solutions (as the ρ_δ are). Since the data $\rho_{0\delta}(x)$ are ordered, so is the sequence of solutions $\rho_\delta(x,t)$,

$$\rho_\delta(x,t) \le \rho_{\delta'}(x,t) \quad \text{if } 0 < \delta < \delta'.$$

By letting $\delta \to 0$ we conclude that there exists a unique limit ρ^*, solution of the target equation with initial data $\rho_0(x) = u_0(x) + v_0(x) \in L^1_+(\mathbb{R})$, such that the mass is conserved (i.e., maximal):

$$\int_{\mathbb{R}} \rho^*(x,t) dx = \int_{\mathbb{R}} \rho_0(x) dx,$$

for all $t \in \mathbb{R}^+$. See Theorem 1 of [6]. It is also proved that function $\rho^*(x,t)$ is positive and C^∞-smooth in Q, and it is bounded for $t \geq \tau > 0$.

In a second step of this case, we consider the limit $\delta \to 0^+$ with ε fixed, and then the limit $\varepsilon \to 0^+$. As long as ε is fixed, we have a local theory with local L^1-dependence (Section 4). Therefore, for all $\varepsilon > 0$ the sequence $\rho_{\varepsilon,\delta}$ admits almost everywhere a limit, ρ_ε, which is the weak solution of System (2) with data ρ_0. By the estimates of Lemma 1,

$$\int_{\mathbb{R}} \rho_\varepsilon(x,t)\,dx = \int_{\mathbb{R}} \rho_{0\varepsilon}(x)\,dx$$

for all $\varepsilon > 0$. We know that ρ_ε is bounded in $L^1 \cap L^\infty(Q_T)$, therefore there exists a subsequence that converges weakly to some ρ_* in $L^2(Q_T)$, $\rho_* \geq 0$.

(ii) To continue the proof in the case of special data, we still have to compare ρ_* with ρ^* and show that they coincide a.e. This happens in three steps:

By the Maximum Principle (see Corollary 11 and Appendix I), we know that the sequence $\rho_{\varepsilon,\delta}$ is monotone, $\rho_{\varepsilon,\delta_1} \leq \rho_{\varepsilon,\delta_2}$ for almost all $(x,t) \in \mathbb{R} \times \mathbb{R}^+$ when $\delta_1 \leq \delta_2$. Hence, $\rho_{\varepsilon,\delta} \geq \rho_\varepsilon$ a.e. for all ε. In the limit $\varepsilon \to 0^+$ we have $\rho_\delta \geq \rho_*$ almost everywhere. Taking now limits as $\delta \to 0^+$, we get $\rho^* \geq \rho_*$ almost everywhere.

Proving equality of those functions needs now a delicate analysis of the L^1 norm of the solutions. Since we know that they are ordered, they if we prove for every t the mass is the same,

$$\int_{\mathbb{R}} \rho_*(x,t)\,dx = \int_{\mathbb{R}} \rho^*(x,t)\,dx,$$

then they must be be a.e. identical. We will estimate separately the mass of ρ^* and that of ρ_*.

One part is easy: following the proof of convergence of $\rho_{\varepsilon,\delta}$ first to ρ_δ and then to ρ^*, we can conclude that $\int_{\mathbb{R}} \rho_\delta(x,t)\,dx = \int_{\mathbb{R}} \rho_0(x)\,dx$, since the limit $\varepsilon \to 0$ is taken under regular conditions so that Corollary 5 applies. On the other hand, the second limit (as $\delta \to 0$ is monotone); it follows that

$$\int_{\mathbb{R}} \rho^*(x,t)\,dx = \int_{\mathbb{R}} \rho_0(x)\,dx.$$

The estimate for ρ_* is based on a technical estimate using the decay assumption (30). In order not to disturb more the presentation at this point, we give the details in Appendix II, see Lemma 13, and continue. The gist of the result is that when we apply it to the functions ρ_ε, we are able to conclude that for some $\beta > 0$ the moments

$$\int_{\mathbb{R}} (1+x^2)^q (\rho_{\varepsilon,\delta} - 2\delta)(x,t)\,dx \leq C$$

with $q = p - (1/2) > 0$ and a bound that is independent of ε, δ and $t \in (0,T)$. This means that for every $\eta > 0$ there is an R such that

$$\int_{|x|>R} \rho_\varepsilon(x,t)\,dx \leq \eta.$$

uniformly in ε, and $t \in (0,T)$. This uniform estimate means that no mass of ρ_ε is lost at infinity when we perform the second limit, $\varepsilon \to 0$ so that $\rho_\varepsilon \rightharpoonup \rho_*$. We conclude that mass is also conserved in the limit

$$\int_{\mathbb{R}} \rho_*(x,t)\,dx = \int_{\mathbb{R}} \rho_0(x,t)\,dx$$

Since $\rho^*(x,t) \geq \rho_*(x,t) \geq 0$ a. e., this means $\rho^*(x,t) = \rho_*(x,t)$ almost everywhere, for all $t > 0$.

By putting $\rho^*(x,t) = \rho_*(x,t) = \rho(x,t)$, we have proved so far, in the case $|\alpha| \leq 1$, that if $(\rho_\varepsilon, j_\varepsilon)$ is a sequence of solutions for the initial value problem of System (4), under the special assumptions on the initial values, then $\rho_\varepsilon(x,t) = u_\varepsilon(x,t) + v_\varepsilon(x,t)$ converges to $\rho(x,t)$, solution of the target equation with initial conditions $\rho(x,0) = u_0(x) + v_0(x)$.

(iii) Let us now improve the mode of convergence and show that $\rho_\varepsilon \to \rho$ in $C([0,T); L^1(\mathbb{R}))$. Firstly, the tail estimates apply to the family $\bar\rho_{\varepsilon,\delta} = \rho_{e,\delta} - 2\delta \geq 0$ uniformly in ε and δ and $0 < t < T$. Since $\bar\rho_{\varepsilon,\delta} \to \rho_\varepsilon$ as $\delta \to 0$, we are thus reduced to prove the convergence in $C([0,T) : L^1(I))$, with $I = (-R, R)$ for all $R > 0$.

Secondly, we estimate the difference $\rho_{\varepsilon,\delta} - \rho_\varepsilon$ by taking into account the conservation of mass

$$\int_{\mathbb{R}} \bar\rho_{\varepsilon,\delta}(x,t)\,dx = \int_{\mathbb{R}} \rho_\varepsilon(x,t)\,dx$$

(recall that $\bar\rho = \rho - 2\delta$ is a solution with regular rate), the ordering, $\rho_{\varepsilon,\delta}(x,t) \leq \rho_\varepsilon(x,t)$ and the decay at infinity. After an easy calculation we conclude that for every $\eta > 0$ there exist R large enough and δ small enough so that

$$\int_{-R}^{R} |\rho_{\varepsilon,\delta}(x,t) - \rho_\varepsilon(x,t)|\,dx \leq \eta,$$

for all $0 < t < T$ and all $\varepsilon > 0$. Hence, we only need to prove the result for $\rho_{\varepsilon,\delta}$ for fixed $\delta > 0$.

In the latter case we prove that the family $\bar\rho_\varepsilon = \rho_{\varepsilon,\delta} - 2\delta$ is uniformly continuous as a function $C([0,T]; L^1(\mathbb{R}))$. Actually, the family $\{\bar\rho_\varepsilon(t)\}$ is relatively compact in $L^1(\mathbb{R})$, uniformly in ε and t, since we have x-regularity coming from the dissipativity of the problem for $|\alpha| \leq 1$ (see Appendix I), which implies that, for all $0 < t$,

$$\|\bar\rho_\varepsilon(x+h,t) - \bar\rho_\varepsilon(x,t)\|_1 \leq \|\rho_0(x+h) - \rho_0(x)\|_1,$$

which tends to zero as $h \to 0$. Uniform estimates in time are given by using the first equation of the system, $\bar\rho_t + j'_x = 0$, plus the L^2 bound for j', which is uniform in ε. Here j' is the flux corresponding to $\bar\rho$.

Putting all these estimates together we conclude that $\bar\rho_\varepsilon$ is uniformly continuous as a function $C([0,T]; L^1(\mathbb{R}))$. The same applies to ρ_ε. We conclude that $\rho_\varepsilon \to \rho$ in $C([0,T]; L^1(\mathbb{R}))$.

By the uniqueness of the limit the whole sequence converges and the convergence takes place also in the a.e. sense.

We recall that the case of $\alpha = 1$ is slightly different, since the Cauchy problem does not possess uniqueness of finite mass solutions. But the property of conservation of mass, inherited by the limit, permits us to identify such a limit with the unique *maximal solution* of (5), characterized by the conservation of mass in the paper [6], to which we refer for more details on the characterization of the maximal solution.

II. *General data of class L^1_+.* We want to remove the special assumptions, namely boundedness for the initial data and the decay at infinity. We proceed as follows. Given a pair of initial functions $(u_0, v_0) \in L^1_+(\mathbb{R})$, for every integer $n > 1$ we construct an approximate initial condition

$$u_{0,n}(x) = \min\{u_0(x), n\}\chi_n, \qquad v_{0,n}(x)) = \min\{v_0(x), n\}\chi_n$$

where χ_n is the characteristic function of the interval $[-n, n]$. We denote by $\rho_{\varepsilon,n}(x,t)$ the first component of the solution for the macroscopic equations corresponding to such initial conditions.

The initial data $u_{0,n}(x)$ and $v_{0,n}(x)$ obviously satisfy the conditions of part (i), hence we deduce that $\rho_{\varepsilon,n} \to \rho_n$ with convergence in $C([0,T];L^1(\mathbb{R}))$.

Let now ρ_ε be the first component of the mild solution for the macroscopic equations corresponding to the initial conditions $u_0(x)$ and $v_0(x)$. Because of the dissipativity of the problem, we have for all $t > 0$

$$\|\rho_{\varepsilon,n}(t) - \rho_\varepsilon(t)\|_1 \leq \int_{|x|>n} (u_0(x) + v_0(x))\,dx + \int (|u_0(x) - n|^+ + |v_0(x) - n|^+)\,dx,$$

and this bound tends to zero as $n \to \infty$ because the initial data are of class L^1. Therefore, for every k there exists $n(k)$ such that for $n \geq n(k)$ we have

$$\|\rho_{\varepsilon,n}(t) - \rho_\varepsilon(t)\|_1 \leq \frac{1}{k}$$

The same contraction estimate applies to the target equation with same initial data, according to the results of [6], hence the same inequality holds with $\rho_{\varepsilon,n}(t)$, $\rho_\varepsilon(t)$ replaced by $\rho_n(t)$, $\rho(t)$, where ρ is the solution of the target equation corresponding to data $u_0 + v_0 \in L^1_+(\mathbb{R})$. Since we have proved in the previous step that $\rho_{\varepsilon,n}$ converges to ρ_n in the sense of $C([0,T];L^1(\mathbb{R}))$, by the triangle inequality the same happens for ρ_ε that tends to ρ in $C([0,T];L^1(\mathbb{R}))$ as $\varepsilon \to 0$.

III. *Convergence of u and v.* The proof is performed first under the conditions of **I** above, passing then to the limit in the general case by approximation and dissipativity with implies continuous dependence for u and v with respect to the initial data uniformly in ε. In case (i) we have the convergence of $u_{\varepsilon,\delta} \to \frac{1}{2}\rho_\delta$, together with the comparison $u_{\varepsilon,\delta} \geq u_\varepsilon$, which gives as $\delta \to 0$ an inequality

$$\limsup_{\varepsilon \to 0} u_\varepsilon \leq \frac{1}{2}\rho,$$

on compact subsets of $Q = \mathbb{R} \times (0,\infty)$. The same holds for v_ε. Since $\rho_\varepsilon = u_\varepsilon + v_\varepsilon$ the lim sup is indeed a limit and we have equality and strong convergence. □

Corollary 13 *The hydrodynamical limit of the previous theorem can be obtained as a double limit of the functions $\rho_{\varepsilon,\delta}$ as $(\varepsilon,\delta) \to (0^+,0^+)$ with no restriction on the sequences (ε,δ).*

The proof relies on the observation that all sequences in the double limit lead to sequences of solutions sandwiched between the two iterated limits.

8.1 Extension. Initial data in the class $L^1 + L^\infty$

We can establish the hydrodynamical limit for the hyperbolic System 2 with rate $k = k_\alpha$ with $|\alpha| \leq 1$ when the initial conditions and solutions belong to larger functional classes. The idea is to use the solutions of already established cases as barriers for the new problems and apply the maximum principle. We consider the space $X = (L^1(\mathbb{R}) + L^\infty(\mathbb{R}))_+$ consisting of all measurable real functions $f \geq 0$, such that we can find a decomposition $f = f_1 + f_2$, with $f_1 \in L^1(\mathbb{R}), f_2 \in L^\infty(\mathbb{R})$. It is equivalent to say that $f \geq 0$ is measurable and there exists an $M > 0$ such that $(f - M)_+ \in L^1(\mathbb{R})$.

The following theorem is an easy extension of Theorem 12:

Theorem 14 *Let $k = k_\alpha$ with $|\alpha| \le 1$, and let $(u_\varepsilon, v_\varepsilon)$ a sequence of mild solutions for the initial value problem of System (2), with nonnegative initial values $u_{0\varepsilon}(x,0)$, $v_{0\varepsilon}(x,0) \in X$, converging to some u_0, $v_0 \in X$ as $\varepsilon \to 0$.*

Then there exists a positive and smooth function ρ such that $\rho_\varepsilon(x,t)$ converges to $\rho(x,t)$ in $L^1_{loc}(Q)$. When $-1 \le \alpha < 1$, the limit density $\rho(x,t)$ is the unique weak solution of the Cauchy problem for the nonlinear heat equation (5) with initial data $\rho_0(x) = u_0(x) + v_0(x)$. If $\alpha = 1$, we obtain the unique unique maximal solution of (5).

Proof: (a) The case of data and solutions in $L^\infty_+(\mathbb{R})$ offers no difficulties, being sandwiched between the already solves cases $u_0, v_0 \in L^1(\mathbb{R})$ on the one hand and $u_0, v_0 \in L^\infty(\mathbb{R})$, $u_0, v_0 \ge \delta > 0$ on the other. We get convergence to the diffusive limit with a rather weak condition: since we know that the family ρ_ε is bounded, we conclude that $\rho_\varepsilon \to \rho$ in the weak* sense of L^∞.

(b) For the case $u_0, v_0 \in L^1(\mathbb{R}) + L^\infty(\mathbb{R})$ we cut the functions at height n (much as we did in part (ii) of Theorem 12, and use part (a) to produce approximations from below. We then use the L^1-contractivity to pass to the limit as $n \to \infty$. It is a bit more of work, since we have to compare solutions. Convergence takes now place in L^1_{loc} weak. □

This result covers in particular data in the spaces $L^p(\mathbb{R})$ for all $1 \le p \le \infty$. According to the estimate of Lemma 1 the solutions would in that case be bounded in $L^\infty((0,T); L^p(\mathbb{R}))$.

9 Other Extensions. Bounded domain with Neumann data

Let us now consider System (2) posed in bounded space interval, say $I = (-R, R)$. Let us add initial conditions as before, with

$$u_{0\varepsilon} \in L^1_+(I), \quad v_{0\varepsilon} \in L^1_+(I),$$

and Neumann conditions

$$u_x + v_x = 0 \quad \text{for } x = \pm R, \ t > 0.$$

This conditions are meant to preserve in time the total mass

$$\int_I (u_\varepsilon(x,t) + v_\varepsilon(x,t))\, dx = \int_I (u_{0\varepsilon}(x) + v_{0\varepsilon}(x))\, dx.$$

We can then re-do the L^∞ theory for regular k's and the L^1 theory for k_α with no major changes, thus establishing the diffusive limit of the hyperbolic system. Note that in this case $L^\infty(I) \subset L^1(I)$.

One immediate application is the diffusive limit in $\mathbb{R}$ with L^1-periodic initial data, i.e. data that are $L^1(I)$ for some interval I as above and repeat periodically outside of I with period $2R$. The problem is equivalent to the previous problem in I with Neumann data.

We leave the easy details to the interested reader.

Appendix I

In this section we study the properties of *dissipativity*, *ω-dissipativity* and *T-dissipativity* for System (2), started in Section 7. Let us start by proving the assertion about the rates $k_\alpha(u, v, x) = (u + v)^\alpha$.

Proposition 15 *The rate* k_α *is* T*-dissipative for* $|\alpha| \le 1$. *For* $\alpha \ge 1$ *it is* ω*-dissipative on bounded solutions* $0 \le u, v \le M$ *with*

$$\omega = (\alpha - 1)\,(2M)^\alpha.$$

In case $\alpha < -1$ k *is* ω*-dissipative on solutions* $u, v \ge N > 0$ *if* $\omega \ge |1+\alpha|(2N)^\alpha$.

Proof: Consider the expression to be checked for ω-dissipativity:

$$\begin{aligned}&[(u_1+v_1)^\alpha\,(v_1-u_1) - (u_2+v_2)^\alpha\,(v_2-u_2) - \omega(u_1-u_2)]\ \text{sign}\ (u_1-u_2) + \\ &[(u_1+v_1)^\alpha\,(u_1-v_1) - (u_2+v_2)^\alpha\,(u_2-v_2) - \omega(v_1-v_2)]\ \text{sign}\ (v_1-v_2)\,.\end{aligned} \tag{31}$$

We want to prove that it is nonpositive for suitably chosen ω, depending possibly on α. If $u_1 > u_2 \ge 0$, $v_1 > v_2 \ge 0$, we have that $\text{sign}\,(u_1-u_2) = \text{sign}\,(v_1-v_2) = +1$ and therefore the sum in the previous equation is equal or less than $-\omega(|u_1-u_2| + |v_1-v_2|)$ so we may take any $\omega \ge 0$. The same result occurs when $0 \le u_1 < u_2$, $0 \le v_1 < v_2$.

We are left with only two cases to study: one is when $u_1 > u_2 \ge 0$ and, at the same time, $0 \le v_1 < v_2$; the second one when $0 \le u_1 < u_2$ and, at the same time, $v_1 > v_2 \ge 0$. Since they are similar, we perform the calculations only in the first case. Then equation (31) reduces to

$$[2(u_1+v_1)^\alpha + \omega]\,(v_1-u_1) - [2(u_2+v_2)^\alpha + \omega]\,(v_2-u_2)\,. \tag{32}$$

We consider now for x, $y \ge 0$, the function

$$f = \{2(x+y)^\alpha + \omega\}(y-x).$$

In view of the relative values of u_i and v_i we want f to increase with y and decrease with x. Since for $x, y > 0$

$$\frac{\partial f}{\partial x} = 2(x+y)^{\alpha-1}\{(\alpha-1)y - (\alpha+1)x\} - \omega, \quad \frac{\partial f}{\partial y} = 2(x+y)^{\alpha-1}\{(\alpha+1)y + (1-\alpha)x\} + \omega,$$

when $\alpha \in [-1, 1]$ it is immediate that $\omega = 0$ implies $f_x \le 0$ and $f_y \ge 0$ as desired.

In case $\alpha > 1$ the function is monotone non-increasing with respect to the variable x and monotone non-decreasing with respect to y for all $0 \le x, y \le M$ if $\omega \ge (\alpha-1)(2M)^\alpha$. The case $\alpha < -1$ is perfectly similar and the condition is satisfied for $\omega \ge |1+\alpha|(2N)^\alpha$ if $u, v \ge N$.

In order to prove T-dissipativity we replace the sign function by sign^+ in formula (31). There is a difference in the case we have examined, $u_1 > u_2$ and $v_1 < v_2$. Now the inequality to be checked is

$$(u_1+v_1)^\alpha\,(v_1-u_1) \le (u_2+v_2)^\alpha\,(v_2-u_2) \le 0. \tag{33}$$

Using the same notation as before and the monotonicity of f for $|\alpha| \le 1$ we have $f(u_1, v_1) \le f(u_2, v_1) \le f(u_2, v_2)$, which gives the result. □

Functional setting

Let us proceed to review the functional setting for the consideration of semigroups generated by dissipative operators. We will proceed in a rather sketchy way since our main goal is to motivate the definition of dissipative interaction rate. For the reader's convenience we start by recalling some definitions. An operator

$$A \ : \ D(A) \subset L^1(X) \to L^1(X)$$

is called *dissipative* if it is closed and if it is such that, for all $f_1,\ f_2 \in D(A)$:

$$\int_X (Af_1 - Af_2)\ \mathrm{sign}\,(f_1 - f_2)\,dx \le 0.$$

Moreover, an operator

$$A \ : \ D(A) \subset L^1(X) \to L^1(X)$$

is called *ω-dissipative* if it is closed and there exists $\omega \in \mathbb{R}^+$ such that, for all $f_1,\ f_2 \in D(A)$:

$$\int_X [(A-\omega I)f_1 - (A-\omega I)f_2]\ \mathrm{sign}\,(f_1 - f_2)\,dx \le 0.$$

Finally, an operator

$$A \ : \ D(A) \subset L^1(X) \to L^1(X)$$

is called *T-dissipative (in the sense of Bénilan)* if it is closed and if it is such that, for all f_1, $f_2 \in D(A)$:

$$\int_X (Af_1 - Af_2)\ \mathrm{sign}^+(f_1 - f_2)\,dx \le 0,$$

where

$$\mathrm{sign}^+(r) \begin{cases} 1 & x \ge 0 \\ 0 & x < 0. \end{cases}$$

If A is dissipative (resp. ω-dissipative, T-dissipative) we say that $-A$ is *accretive* (resp. *ω-accretive*, *T-accretive*). Dissipative operators generate semigroups of contractions in $L^1(\mathbb{R})$ by solving the abstract equation $f_t = Af$, $f(0) = f_0$. The range of data that may serve as initial data is characterized in terms of a range condition for the operator. The solutions are called mild solutions. This is well-known theory as developed by Bénilan, Crandall and other researchers, cf. [2].

Let us apply these concepts to our problem. We have to investigate the behavior of the operator

$$B_k f = \left(-\frac{\partial u}{\partial x} + k(u,v,x)(v-u), \frac{\partial v}{\partial x} + k(u,v,x)(u-v) \right) =$$

where $f = (u,v)$. We split the operator in two parts

$$\frac{\partial}{\partial x}(-u,v) + A_k f.$$

We write B_α, A_α instead of B_k, A_k when k is replaced by k_α. A_α is the operator defined by

$$A_\alpha f = \left(\rho^\alpha(v-u), \rho^\alpha(u-v)\right),$$

with $\rho = u + v$. It is well-known that the linear operator

$$C_\alpha = \frac{\partial}{\partial x}(-u, v)$$

is T-dissipative in $L^1(\mathbb{R})$ with domain $W^{1,1}(\Omega) \times W^{1,1}(\Omega)$. The main point of the introduction of the definitions of dissipativity for interaction rates is the following result.

Lemma 11 *A_k is a dissipative operator in $L^1(\mathbb{R})$ (resp. T-dissipative, ω/ε^2 dissipative) if it is closed and k is a dissipative rate (resp. T-dissipative, ω dissipative).*

As is the rule with unbounded operators, the domains have to be carefully defined. For $k = k_\alpha$ the domain of A_α is taken as $D^+(A_\alpha)$, i.e. the subset of $(u, v) \in L^1_+(\mathbb{R}) \times L^1_+(\mathbb{R})$ such that $k_\alpha(u, v)(v - u) \in L^1(\mathbb{R})$. For $\alpha > 1$ we take

$$D^*(A_\alpha) = \{(u, v) \in D^+(A_\alpha) : \|u\|_\infty, \|v\|_\infty < \infty\},$$

and for $\alpha < -1$ we take the domain $D^+_\delta(A_\alpha) \subset D^+(A_\alpha)$ where $u, v \geq \delta > 0$.

After some calculations we can get

Lemma 12 *Let $-1 \leq \alpha \leq 1$. Then the operator B_α is T-dissipative from the domain $D^+(B_\alpha) = D^+(A_\alpha) \cap (W^{1,1}(\Omega) \times W^{1,1}(\Omega))$ into $L^1_+(\Omega) \times L^1_+(\Omega)$.*

Standard theory, cf. [1], allows now to prove the contraction theorem.

Corollary 16 *Let $f_1 = (u_1(x, t), v_1(x, t))$ and $f_2 = (u_2(x, t), v_2(x, t))$ two solution of System (2) when $|\alpha| \leq 1$, with initial data of class L^1 $f_{0,1} = (u_{1,0}(x), v_{1,0}(x))$ and $f_{0,2} = (u_{2,0}(x), v_{2,0}(x))$ respectively. Then*

$$\|f_1 - f_2\|_1 \leq \|f_{0,1} - f_{0,2}\|_1.$$

Moreover, if

$$u_{1,0}(x) \leq u_{2,0}(x) \quad \text{and} \quad v_{1,0}(x) \leq v_{2,0}(x)$$

almost everywhere, then the solutions are such that $u_1(x, t) \leq u_2(x, t)$ and $v_1(x, t) \leq v_2(x, t)$ almost everywhere, for all $t \in \mathbb{R}^+$.

Appendix II. Tail control

We will obtain decay estimates for solutions of the hyperbolic system with a rate k such that for some $c > 0$ and $\alpha \geq 0$

$$k(u, v, x) \geq c(u + v)^\alpha. \tag{34}$$

holds for all $u, v \geq 0$. The estimates are useful in controlling the tails of the solutions (i.e., the small mass as $|x| \to \infty$) when $\alpha < 2$.

Lemma 13 *Let u_ε and v_ε be the solutions of System (2), with initial data $u_0(x), v_0(x) \in L^\infty_{loc}(\mathbb{R})$, $u_0(x), v_0(x) \geq 0$. Let k be admissible and satisfy the previous growth condition. We assume that the initial data satisfy*

$$\int (u_0(x) + v_0(x))^2 (1 + |x|^2)^p \, dx < C_0, \tag{35}$$

for some $p < (2/\alpha) - (1/2)$*. Let* $\rho_\varepsilon = u_\varepsilon + v_\varepsilon$ *and* $j_\varepsilon = (u_\varepsilon - v_\varepsilon)/\varepsilon$*. Then,*
(i) for every $0 < t < T$

$$\int \rho_\varepsilon^2(x,t)\,(1+|x|^2)^p\,dx < C_T. \tag{36}$$

(ii) We also have the bound:

$$\int_0^T \int_{\mathbb{R}} j_\varepsilon^2(x,t) k(u_\varepsilon, v_\varepsilon, x)(1+|x|^2)^p\,dxdt \le C_T. \tag{37}$$

(iii) Finally, we get the following estimate for L^1*-integrability with a weight:*

$$\int \rho_\varepsilon(x,t)\,(1+|x|^2)^{q/2}\,dx < C_T. \tag{38}$$

for $q = p - 1/2 < (2/\alpha) - 1$*.* C_T *denotes a positive constant that depends on* T *and the bounds for* k *and the initial data mentioned in the statement, but does not depend on* ε*.*

Proof: It is variant of previous results of this paper. We multiply the two equations of System (2) by $u_\varepsilon\phi^2$ and $v_\varepsilon\phi^2$ respectively, for some smooth test function $\phi(x)$ to be precisely chosen below. Adding the obtained equations and integrating we get

$$\frac{d}{dt}\int_{\mathbb{R}} \left[u_\varepsilon^2 + v_\varepsilon^2\right]\phi^2 dx + 2\int_{\mathbb{R}} k_\varepsilon j_\varepsilon^2 \phi^2\,dx = \frac{2}{\varepsilon}\int_{\mathbb{R}} (u_\varepsilon^2 - v_\varepsilon^2)\phi\phi_x\,dx.$$

with $k_\varepsilon = k_\varepsilon(u_\varepsilon, v_\varepsilon, x)$. The last term can be written as $I = 2\int j_\varepsilon(u_\varepsilon + v_\varepsilon)\phi\phi_x\,dx = 2\int j_\varepsilon\rho_\varepsilon\phi\phi_x\,dx$. It can be split into

$$I \le \int_{\mathbb{R}} k_\varepsilon j_\varepsilon^2\phi^2\,dx + \int_{\mathbb{R}} \frac{\rho_\varepsilon^2}{k_\varepsilon}\phi_x^2\,dx.$$

Using the growth condition we know that $\rho_\varepsilon^2/k_\varepsilon \le c\,\rho_\varepsilon^{2-\alpha}$, hence the last term is bounded by

$$\left(\int_{\mathbb{R}} \rho_\varepsilon^2\phi^2\,dx\right)^{1-(\alpha/2)} \left(\int_{\mathbb{R}} \phi_x^{4/\alpha}\phi^{2-(4/\alpha)}\,dx\right)^{\alpha/2},$$

The last factor is bounded for the choice $\phi(x) = (1+|x|^2)^{p/2}$ with $p < (2/\alpha) - (1/2)$. Writing $X(t) = \int_{\mathbb{R}} \left[u_\varepsilon^2(x,t) + v_\varepsilon^2(x,t)\right]\phi^2 dx$, we arrive at the inequality

$$\frac{dX(t)}{dt} + 2\int_{\mathbb{R}} k_\varepsilon j_\varepsilon^2\phi^2\,dx \le C_p\,X^{1-(\alpha/2)}.$$

Disregarding the intermediate term and integrating from 0 to $t \in (0,T)$, we get the boundedness of $X(t)$ for all $t \le T$ with constant C_T. This proves parts (i) and (ii).

(iii) Using Hölder's inequality we may translate the previous square integrability into plain integrability. Indeed, if $f \ge 0$ is a locally integrable real function such that $\int f(x)^2(1+|x|^2)^p\,dx < C$, then

$$\int f(x)(1+|x|^2)^{q/2}\,dx$$

is bounded if $q < p - 1/2$. □

Remarks (1) Whenever $\alpha < 2$ we can choose some $p > 1/2$ in assumption (35). The result applies with a $q > 0$, and then estimate (38) means uniform smallness of the space integral of ρ_ε for all $|x| \geq R$ with R large, and all $0 < t < T$, i.e., a uniformly small tail.

(2) On the other hand, similar tail estimates for $\alpha \geq 2$ are not to be expected, since they have been proved to be false for the diffusive limit, equation (6), with $D(\rho) = \rho^{-\beta}$, $\beta \geq 2$. Indeed, solutions of this equation with integrable initial data do not exist, and an initial discontinuity layer is formed. When we perform the approximation with lifted data and pass to the limit the mass just disappears at infinity.

(3) In the application it may be true that condition (34) holds only for values $0 \leq u, v \leq M$. Then the conclusion applies only to bounded solutions with values in that range. In that case $0 \leq \rho_\varepsilon \leq C$ and the result holds even for $\alpha < 0$ (with p and q as in the case $\alpha = 0$).

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