OPINION DYNAMICS: KINETIC MODELLING WITH MASS MEDIA, APPLICATION TO THE SCOTTISH INDEPENDENCE REFERENDUM

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Abstract. We consider a kinetic model describing some mechanisms of opinion formation in the framework of referendums, where the individuals, who can interact between themselves and modify their opinion by means of spontaneous self-thinking, are moreover under the influence of mass media. We study, at the numerical level, both the transient and the asymptotic regimes. In particular, we point out that a plurality of media, with different orientations, is a key ingredient to allow pluralism and prevent consensus. The forecasts of the model are compared to some surveys related to the Scottish independence referendum of 2014.

1. Introduction

The present work provides the study of some phenomena arising in social sciences by means of a statistical mechanics approach. This strategy was born in the eighties; the reader can check [14] and the references therein to know more about the topic. In particular, one can find there some discussions about the French referendum on the European constitution in 2005.

Forecasting the opinion evolution with respect to a binary question is crucial in many situations. A typical example consists in the anticipation of a referendum result or an electoral competition, by using poll data from surveys held some time before the vote.

In this article, we give a contribution to this problem by studying a mathematical model based on a kinetic approach, and we provide both qualitative and quantitative comparisons with real data, in the case of the 2014 Scottish independence referendum.

Our model is based on the following hypotheses.

Assumption 1. The number of individuals in the population is constant. Of course, this is only relevant for short-term forecasts, as during a referendum campaign.

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Assumption 2. The community is fully interconnected and all the members can somehow discuss with each other. By tuning the cross section of the model, however, it is possible to generalize our model to situations where the interaction probability between individuals depends on their own convictions, as in bounded confidence models [10, 15].

Assumption 3. There are some mass media which can have an effect on the individuals.

Let us focus on the mass media, whose involvement is the most significant contribution of this work. Although the press freedom appears as a safeguard in a democracy [19], the media can also be used to influence the public opinion, by taking advantage of their possibility of easily reaching a wide audience and favour some interests, ideas or arguments inside a population [9]. In order to influence the population, many different tactics have been developed. Among them, we quote the distraction, in different forms (which is based on the assumption that the public has a limited attention span), the appeal to consensus and the fear mongering [13]. These techniques are used either separately, or combined between them, and can be applied to many different contexts. Mass media use diversified media technologies in order to reach a large audience and transfer concepts, ideas, *etc.* Among them, we can cite broadcast media (radio, television), print media (newspapers, books), outdoor media (posters), digital media (Internet, mobile mass communication).

Mass media can use two different kinds of strategy. They can act either as observers, or as opinion carriers. In the latter case, they often simply spread out the opinions which they support into the population, without any other interest than the maximisation of their audience. Unfortunately, these two strategies cannot be modelled in the same way. Manipulating media can be described as entities with a given external opinion, which aim to spread their own opinion inside the population. Wide audience media should be described with a more intricate approach: the opinions they carry can depend on the opinion distribution inside the population, and then the influence of the population on the media opinion is a part of the model itself. In this work, we only consider media with an *a priori* given opinion. This case can in fact be seen as a first rough approximation of the wide audience media, where the media opinion may not remain constant, but does not depend on the opinion distribution.

We here use a kinetic approach to describe and forecast the evolution of a system under the aforementioned effects. In this framework, a distribution function holds the information on the system, and its time evolution is governed by a partial differential equation with integral operators. This strategy is based on sophisticated mathematical tools and its interest is apparent when the number of individuals becomes very large, since it allows to handle collective behaviours.

Kinetic equations have been used to model social phenomena since the early 90s, when Helbing studied behavioural changes by using Boltzmann-like equations [17, 16, 18]. Subsequently, this approach has been the basis of several works, see the review article [8] and the references therein. Note that, in most part of the literature based on all the approaches, including the kinetic one, the key phenomenon is compromise, see [1, 2, 20, 12, 4].

The model we here investigate also owns this binary interaction feature: individuals follow the rule proposed by the authors in [7]. Taking into account the influence of mass media on the population, we are able to improve our previous results from [7], and recover some realistic behaviours.

The opinion variable of our model runs in a continuous way between two extreme values, from -1 to $+1$. The model describes the time evolution of a distribution function f (depending on both opinion and time), which represents the density of individuals with respect to the opinion about the binary question. That kind of variable reflects the opinion formation process, which does not lead necessarily to strong opinions, since doubt and partial agreement are often predominant feelings.

The media action on the system is modelled by a kinetic operator which has a structure similar to the media operator introduced in [6], but with a different nonlinear effect in the post-interaction opinion. The structure of this linear (with respect to f) term is well adapted to the coupling with the self-thinking.

At the end of the opinion formation process, the opinions must be translated into a decision. This issue is not tackled here, since it is often the result (at least in the case of reasonable and rational individuals) of a game-theoretical approach, especially when those individuals have intermediate opinions.

Since we work with a continuous opinion variable, it is difficult, in general, to get comparisons with real data. Indeed, polls usually have a binary ("yes"/"no") or ternary ("yes"/"no"/"indifferent") structure, since they are interested in forecasting the result of the final choice with respect to the binary question of the referendum. However, some more structured surveys have been built on a more complete scale (usually from 1 to 10). These polls can be a good tool for comparisons, which are essential to somehow validate our model. The last part of our study is consequently devoted to the qualitative comparison between the results of our model and three surveys performed by the polling corporation *ICM Unlimited*[1] about the Scottish independence referendum, that took place in Scotland on September 18, 2014. Note that the data-supported profile of f seems stable with respect to the opinion variable. The data allow to fit some mechanical parameters, but they do not provide any information related to the time scales: an answer to this question would require the knowledge of the same kind of data in a wider range of time.

[1]See `http://www.icmunlimited.com/media-centre/polls/`

The article is organized as follows. In the next section, we describe the model and its structure. Numerical simulations are performed and commented in Section 3. Eventually, Section 4 focuses on the Scottish independence referendum: we relate our numerical results to some existing polls obtained before the referendum itself. It provides some kind of validation to our model, and already suggests possible improvements.

2. Kinetic model

In this section, we briefly describe the model we investigate here. As already stated, it appears as an extension to the one previously introduced in [7]. We refer to that article for a full description of the bilinear integral operator defining the binary interaction between the agents.

In what remains, Ω denotes the open interval $(-1,1)$. The variables of the model are opinion x, a continuous variable belonging to $\bar{\Omega} = [-1,1]$, and time $t \in \mathbb{R}^+$. The opinion variable x describes the degree of agreement with respect to a binary question, for example, a referendum. In particular, $x = \pm 1$ are the two extreme answers to the question, *i.e.* "yes" or "no" without reserve, whereas any intermediate value between -1 and $+1$, 0 excluded, means that the corresponding agent partially agrees with the opinion labelled with the same sign, with a degree of conviction which is proportional to $|x|$. The value $x = 0$ means that the agent is undecided.

The community is described by means of the distribution function $f := f(t,x)$, defined on $\mathbb{R}_+ \times \bar{\Omega}$, whose time evolution is governed by a kinetic equation. The main ingredients of the kinetic model act at two different levels. The first one is the description of the microscopic active phenomena, in this case, the self-thinking, the binary opinion exchange between the individuals of the population and the effect of mass media on a single individual. The second level governs the time evolution of the distribution function, which is induced by the operators which take the microscopic phenomena into account. Since the model is of kinetic type, we borrow the language of kinetic theory. Hence, the term *collision* means an interaction with exchange of opinions, that gives, as a result, a modification of the agents opinions.

Self-thinking. The self-thinking phenomenon is described by a diffusion operator obeying to a non-uniform Fourier law, with Fourier term $\alpha = \alpha(x)$. This term quantifies the possibility that people may change their opinion through personal reasoning. In particular, we assume that the Fourier term $\alpha : \bar{\Omega} \to \mathbb{R}$ is a nonnegative C^1 function such that $\alpha(-1) = \alpha(+1) = 0$. This last constraint ensures, from the modelling viewpoint, that the opinions cannot go out the interval $\bar{\Omega}$.

Binary opinion exchanges. Let x, $x_* \in \bar{\Omega}$ the opinions of two individuals of the population before interacting, and x', $x'_* \in \bar{\Omega}$ the corresponding opinion after the binary exchange.

We assume that the individuals of the population are of conciliatory type [7, 5]: it means that they have a natural trend to reach a compromise with each other. From a quantitative point of view, the opinions after interaction are modified in order to get closer to the average opinion before the interaction and, at the same time, stronger opinions are less attracted towards the average than weaker ones. More precisely, the collision rule adopted in the article is the following:

$$x' = \frac{x + x_*}{2} + \eta(x)\,\frac{x - x_*}{2}, \qquad x'_* = \frac{x + x_*}{2} + \eta(x_*)\,\frac{x_* - x}{2}, \tag{1}$$

where the *attraction function* $\eta : \bar{\Omega} \to \mathbb{R}$ is C^1, with $0 \leq \eta(x) < 1$ and $\eta'(x) > 0$ for all $x > 0$. It ensures that stronger opinions are less influenced than weaker ones through a binary interaction.

Remark 2.1. *We assume that the Jacobian $J(x, x_*)$ of the collision mechanism* (1) *satisfies $J(x, x_*) \geq J_0$ for any x, $x_* \in \Omega$, for a given $J_0 > 0$. Jacobian $J(x, x_*)$ should indeed be lower-bounded by a positive constant J_0 in order to allow the weak formulation of our problem which is discussed in Appendix A.*

Interaction with mass media. In the model, as it is suggested by Assumption 3, we take into account the existence of m mass media, $m \geq 1$. They can influence the population by sharing their opinion about the referendum. The effects of the media are modelled thanks to an interaction with a given background. To define the characteristics of each media M_j, $1 \leq j \leq m$, we need three quantities: its time-depending strength $\theta_j : \mathbb{R}^+ \to \mathbb{R}^+$, its time-depending opinion $X_j : \mathbb{R}^+ \to \bar{\Omega}$, and its attractiveness $q_j : \bar{\Omega}^2 \to \mathbb{R}$.

The individuals of the population are still considered as conciliatory regarding the media. The evolution of an individual's opinion follows a rule, similar to the binary collision mechanism defined in the previous subsection, but with the difference that the media opinion is not influenced by the population opinion. The main idea of the collision mechanism consists in supposing that the opinions after the interaction are modified in order to approach the average opinion before the interaction and, again, that stronger opinions are less attracted towards the media opinion than weaker ones. Hence, if we denote by $\bar{x}_j$ the post-collisional opinion of an individual of pre-collisional opinion x, for each mass media M_j, $1 \leq j \leq m$, we write

$$\bar{x}_j = \frac{x + X_j(t)}{2} + q_j(x, X_j(t))\frac{x - X_j(t)}{2}, \tag{2}$$

where, for any j, the media attractiveness q_j is continuous on $\bar{\Omega}^2$, and satisfies $0 \leq q_j(x, y) \leq 1$ for any x, $y \in \bar{\Omega}$.

The nonnegativity of q_j is crucial to ensure that the post-collisional opinion $\bar{x}_j$ lies in $\bar{\Omega}$. Moreover, it seems reasonable, from the modelling point of view, to impose that the media attractiveness is close to 1 when $|x| \simeq 1$:

it really means that individuals with a strong opinion are less influenced by the media effects than agents with a weaker opinion.

Time evolution of f. The model governing the evolution of the system is a partial differential integral equation, which is more naturally written in a weak form. Indeed, the existence of a pre-collisional opinion pair generating a post-collisional opinion pair $(x, x_*) \in \bar{\Omega} \times \bar{\Omega}$ through the collision rule (1) is not guaranteed, in general. The weak form ensures that such pathological pairs do not enter into the formulation of the model.

The model hence reads as follows. Let $T > 0$ and $f^{\text{in}} \in L^1(\Omega)$ a nonnegative function. The unknown f satisfies, for almost every $t \in [0, T]$, and in a distributional sense in t,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega f\,\varphi\,\mathrm{d}x\right) = \int_\Omega \partial_x\left(\alpha(x)\partial_x\varphi\right)\, f\,\mathrm{d}x + \langle Q(f,f),\varphi\rangle + \sum_{j=1}^m \langle L_j(f),\varphi\rangle \tag{3}$$

for any test-function $\varphi \in C^2(\bar{\Omega})$, with initial condition

$$f(0,x) = f^{\text{in}}(x), \qquad x \in \bar{\Omega}. \tag{4}$$

Of course, (3) can be written under the classical form

$$\partial_t f = \partial_x(\alpha(x)\partial_x f) + Q(f,f) + \sum_{j=1}^m L_j(f),$$

which must be understood in the distributional sense in both variables (t, x).

Operator Q translates the effects of the binary interactions between the agents, and can be written in different forms. A key ingredient of this bilinear (with respect to f) operator is the cross-section. This nonnegative quantity, denoted by β, governs the probability that an exchange of opinions can occur. We here assume that β is a positive constant: this is the simplest possible assumption, and β is then the collision frequency. Note that β, as a constant, does not depend on $x - x_*$. Consequently, Assumption 2 is satisfied.

The weak form used in this work is

$$\langle Q(f,f),\varphi\rangle = \frac{\beta}{2}\iint_{\Omega^2} f(t,x)f(t,x_*) \left[\varphi(x') + \varphi(x'_*) - \varphi(x) - \varphi(x_*)\right]\mathrm{d}x_*\,\mathrm{d}x. \tag{5}$$

Note that the collision rule (1) only appears in the arguments of the test-function, and the signs in front of each term involving the test-function is in a clear agreement with the usual gain/loss structure of Boltzmann-type operators.

Operators L_j, $1 \le j \le m$, linearly depend on f, and describe the media effect on the population, and they have the following form:

$$\langle L_j f,\varphi\rangle = \theta_j(t)\int_\Omega f(x)\left(\varphi(\bar{x}_j) - \varphi(x)\right)\mathrm{d}x. \tag{6}$$

Note that, in the same way as Q, the interaction rule (2) only appears in the arguments of the test-function, and the signs in front of each term involving the test-function again show a gain/loss structure.

We must emphasize that the weak formulation is required not only to recover the conservation of the number of individuals in the population (by setting $\varphi \equiv 1$), which was required by Assumption 1, but also to obtain an existence result for (3), see Appendix A for more details.

3. Numerical tests

The architecture of the numerical method used in the simulations separately treats the kinetic part and the diffusion term by means of a splitting technique, which was already used in [7]. Both steps are performed by using a regular subdivision $(x_0, \dots, x_N)$ of $\bar{\Omega}$, with $N \geq 1$, and by computing f at the centre $x_{i+1/2}$ of each interval $[x_i, x_{i+1}]$, $0 \leq i \leq N-1$. Let us also set $\Delta x = 2/N$.

In order to represent f in terms of macro-particles, we sample it at each time step on the regular grid, and its evolution is obtained through a slightly modified Bird method [3], which takes into account the collision rules (1). The diffusion part is explicitly treated, and a stability condition is accordingly taken into account. The mass media are a set of invariant particles which carry the media opinion and interact with the distribution function f, chosen by a sample over the whole set of macro-particles, by obeying to the collision rules (2).

The scheme itself conserves the total agents number, *i.e.* $\|f(t)\|_{L^1_x} = \|f^{\text{in}}\|_{L^1}$, and guarantees that the opinion bounds are not violated: indeed, opinions x such that $|x| > 1$ are not possible in both kinetic and diffusive steps of the splitting procedure.

The numerical code has been written in `C`.

In all the computations, the Fourier term is $\alpha(x) = 0.05(1-x^2)^{1/3}$, the collision frequency β is set to 50, and the attraction function is given by $\eta(x) = 0.25(1+x^2)$. The choice of media attractivenesses q_j is crucial. In our tests, we consider two different types of media influence: a global media influence function and a media influence which equals 1 outside an interval $[-\delta, \delta]$, $0 < \delta < 1$. This last choice translates the idea that strong opinions are not affected by the influence of the media, whereas weaker opinions are submitted to the effect of mass media.

In what follows, we investigate test cases to give an overview of the quantitative features of the model. First of all, we study the influence, on a population, of a unique media with a fixed opinion, and two different initial conditions: a balanced initial datum and an unbalanced one. Subsequently, we discuss the interactions between two groups of balanced media, again with fixed opinions. This last result is compared in Section 4 to the opinion dynamics for the Scottish independence referendum.

3.1. **Effect of a unique media on the opinion formation.** The first example studies the behaviour of the initial value problem (3)–(4), with $m = 1$. In all tests of this subsection, we suppose that θ_1 is constant, equal to 50. The media attractiveness is also constant: $q_1 = 0.5$. In Figure 1, we can check the asymptotic states of the system, where the media opinion $X_1(t)$ is set to -0.9, and for both initial data $f_1^{\text{in}} = 0.5$ and $f_2^{\text{in}} = \mathbf{1}_{[0,1]}$.

We note that the asymptotic states are the same, and of course do not depend on initial data. They are driven by the media opinion, even if the population has an initial opinion with opposite sign with respect to the media opinion. Moreover, both situations show that self-thinking prevents the system from a complete adherence to the media opinion.

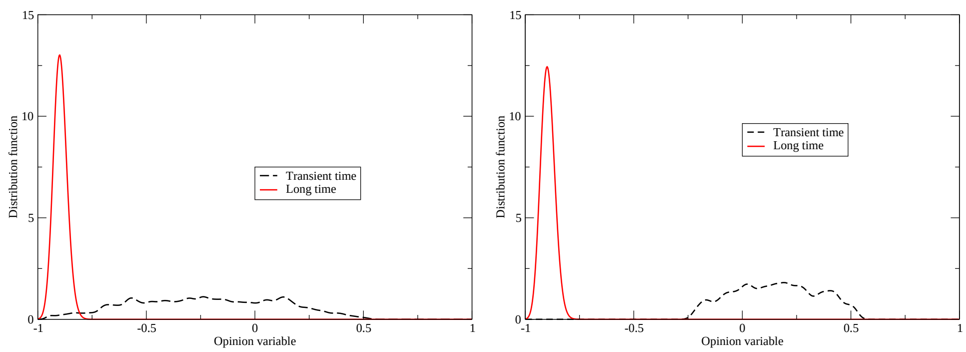

FIGURE 1. Effect of a unique media for (a) $f_1^{\text{in}} = 0.5$, (b) $f_2^{\text{in}} = \mathbf{1}_{[0,1]}$.

In Figure 2, we can see the sensitivity of the model with respect to the media attractiveness q_1. We here choose a regularized version of

$$q_1(x, X_1) = 1 - \frac{1}{2}\mathbf{1}_{[-0.5,0.5]}(x).$$

Note that, since we are working on a discrete grid, we can numerically identify the discretization of q_1 and the discretization of its regularized version, if Δx has a smaller order of magnitude than the regularization parameter.

This media attractiveness means that only individuals with a weak opinion, in the range $[-0.5, 0.5]$, are influenced by the media opinion.

In order to compare the results with the previous case, we choose the same set of initial datum and media opinion as in the second example, *i.e.* $f_2^{\text{in}} = \mathbf{1}_{[0,1]}$ and $X_1(t) = -0.9$.

After a transient period of time, the media has, again, enough strength to drive the whole population towards opinions whose sign agrees with its opinion, even if 50% of the population cannot initially be influenced by the media opinion. This behaviour is due to the binary interactions between the agents. Since the whole population is composed of conciliatory individuals, the effect of the binary interactions leads to compromise. Hence, the fraction

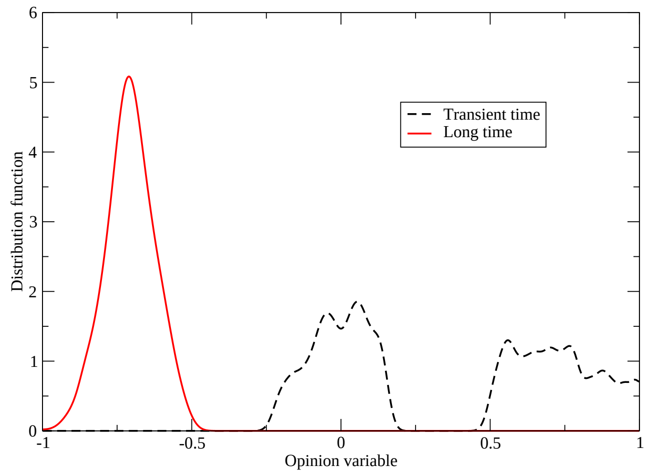

FIGURE 2. Effect of a unique media with finite influence range.

of the population which keeps an opinion outside $(0.5, 1]$ decreases towards 0. On the other hand, since the agents with opinions inside the interval $[-1, -0.5)$ are not influenced by the media, the peak of the distribution is not centred at the media opinion any more, and is closer to 0.

3.2. **Interaction between two media with fixed opinions.** We conclude by pointing out how a plurality of media is a factor that allows pluralism of opinions and the splitting of the population in two well-defined groups. In order to recover this behaviour, we now choose $m = 2$, $\theta_1 = \theta_2 = 50$, and $X_1(t) = -X_2(t) = 0.9$.

In Figure 3, we plot the asymptotic state of the distribution function, with initial datum $f_1^{\text{in}} = 0.5$ and both media attractivenesses given by $q_j = 1$ if $|x - X_j| > 1$ and $q_j = 0.5$ otherwise, for $j = 1, 2$.

After a transient period, we note the formation of two peaks centred at each media opinion ± 0.9. The fraction of agents with a weak opinion is very low. It is a straightforward consequence of the fact that all individuals of the population are under the media influence.

In Figure 4, we plot the asymptotic state of the distribution function, with the same initial datum as in Figure 3, namely $f_1^{\text{in}} = 1/2$. The media attractivenesses are given by $q_j = 1$ if $|x - X_j| > 0.5$ and $q_j = 0.5$ otherwise, for $j = 1, 2$.

In this case, the individuals with opinion close to 0 are not influenced by any media: they can change their opinions only through self-thinking and binary interactions.

We observe the formation of two strong sub-groups with asserted opinions, but also the emergence of a fraction of undecided persons (who may have a higher propensity to abstention). This behaviour is typical of some referendum campaigns, such as the case discussed in the next section. Note that

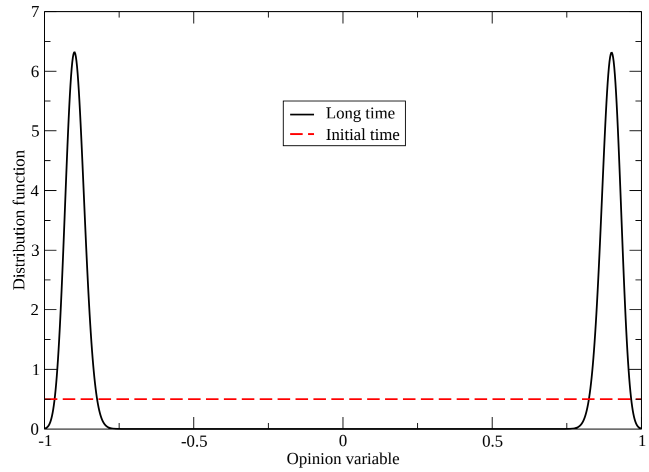

FIGURE 3. Pluralism of media and pluralism of opinions: global media influence

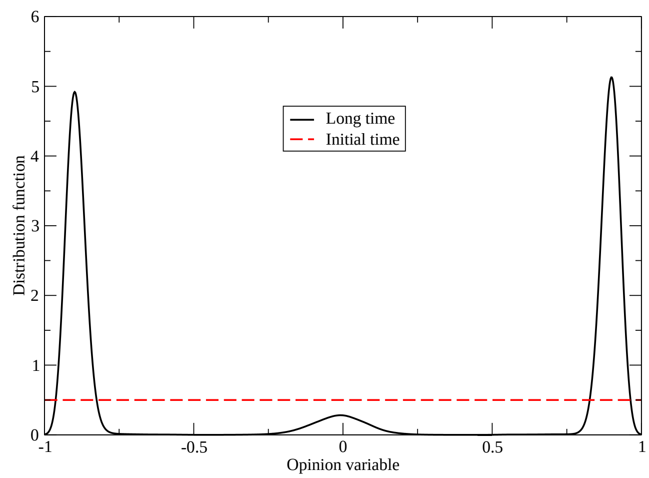

FIGURE 4. Pluralism of media and pluralism of opinions: local media influence.

the observed concentration phenomenon may be analysed with techniques similar to those presented in [11].

4. Comparison with polls: the Scottish case

On September 18, 2014, Scotland was called to answer the independence referendum question: *"Should Scotland be an independent country?"*. The turnout of 84.6% was the highest one recorded for a referendum in the United Kingdom since the introduction of universal suffrage. This datum is enough, by itself, to prove the importance that Scotland attributed to the referendum question. The official results, certified by the Chief Counting

Officer, from the Electoral Management Board for Scotland (EMB), are the following:

Yes	1,617,989	44.65%
No	2,001,926	55.25%
Valid votes	3,619,915	99.91%
Invalid or blank votes	3,429	0.09%
Total votes	3,623,344	100.00%

The referendum was preceded by hundreds of surveys, whose aim was the prediction of the final result. Those surveys took into consideration many different aspects related to the independence referendum question. Among them, we have considered three surveys for the validation phase, which have the advantage of offering multiple answers. They hence allow the comparison with our model, based on a continuum of opinions. The precise data of these polls are the following:

(1) July 11, 2014: *ICM* survey for *Scotland on Sunday* with 1002 respondents, aged 16 and older;
(2) August 13, 2014: *ICM* survey for *Scotland on Sunday* with 1005 respondents, aged 16 and older;
(3) September 16, 2014: *ICM* survey for *The Scotsman* with 1175 respondents, aged 16 and older.

The question was identical in the three polls: *"Can you please say where you are on this scale regarding Scotland becoming independent?"*. The answers were modulated on a scale, between 1 and 10, where

- 1 means "completely against an independent Scotland",
- 5 means "neither for nor against" (positioned between 5 and 6),
- 10 means "completely for an independent Scotland".

The results of the three surveys are all plotted in Figure 5.

As the referendum day approached, there was a clear bipolarization of the population and the emergence of two opposite fields with approximately the same magnitude. The number of undecided and agents with mild opinion decreased, and the media widely spread the arguments of both opposite parties.

The qualitative agreement between the surveys history and the results in Figure 4 is quite good: both peaks are centred at the extreme values (or very close to them). We recognize moreover the existence of some undecided people, which have been reported in the polls (there is a local maximum around the value 5).

The main differences between our simulations and the three surveys is given by the behaviour of moderately decided individuals (located in states 2 to 4 and 7 to 9). Indeed, the polls report that a small, but not negligible, fraction of polled people is located in these intermediary states, whereas, in our numerical simulation, the fraction of the population far from the peaks is very small. However, it is difficult to draw some conclusions from this

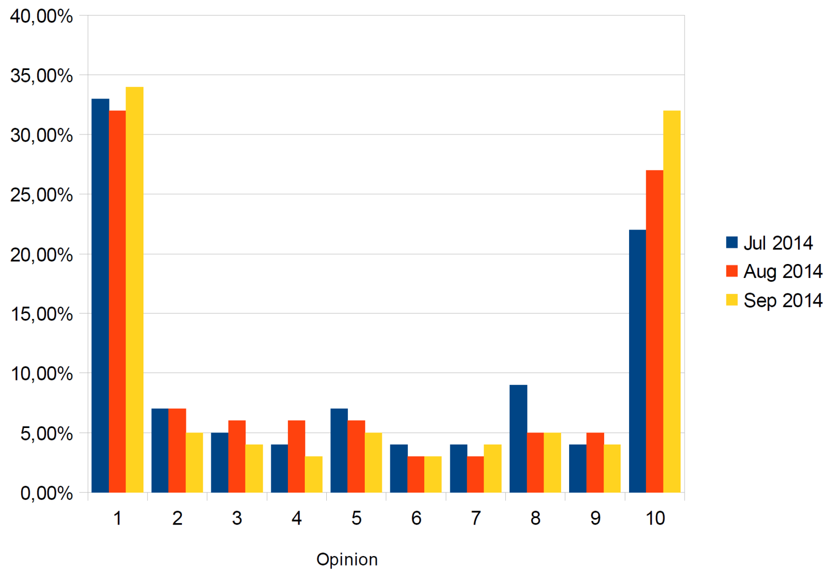

FIGURE 5. Results of three *ICM* surveys.

consideration since the magnitude of the population in each of the intermediary states is often below 5%, and therefore it is quite close to the survey error margin.

5. Conclusion

We presented a kinetic model to describe an interconnected population which must choose an option in the framework of a binary question. The agents can modify their opinion by means of spontaneous self-thinking, discussions with each other, and influence of mass media. After a numerical study, we provided comparisons between the model and the results of three surveys on the Scottish independence referendum of 2014 with a good qualitative agreement. We emphasize that the emergence of clusters is not related to the bounded confidence in the population, as in [10], but occurs thanks to the combined action of the media and the natural tendency to compromise of the agents.

Appendix A. Mathematical properties

Weak formulations are a very convenient tool to deduce some basic mathematical properties, such as the conservation of the number of individuals in the population.

Proposition A.1. *Let* $f = f(t, x)$ *be a nonnegative weak solution of* (3)–(4)*, with a nonnegative initial datum* $f^{\mathrm{in}} \in L^1(\Omega)$*. Then we have*

$$\|f(t,\cdot)\|_{L^1(\Omega)} = \|f^{\mathrm{in}}\|_{L^1(\Omega)} \qquad \text{for a.e. } t \geq 0.$$

Proof. This is a straightforward consequence of (3)–(6), with test-function $\varphi \equiv 1$. □

Some mathematical properties of the equation are a consequence of the structure of the bilinear term. The following result holds, the proof of which is given in [7].

Lemma A.2. *Let* $Q(f,f)$ *defined by* (5) *and*

$$\langle Q^+(f,f),\varphi\rangle = \frac{\beta}{2}\iint_{\Omega^2} f(t,x)f(t,x_*)\left[\varphi(x')+\varphi(x'_*)\right]\mathrm{d}x_*\,\mathrm{d}x.$$

If $f(t,\cdot)\in L^1(\Omega)$, *then both* $Q^+(f,f)(t,\cdot)$ *and* $Q(f,f)(t,\cdot)$ *belong to* $L^1(\Omega)$, *and we have, for a.e.* $t>0$,

$$\|Q^+(f,f)(t,\cdot)\|_{L^1(\Omega)} \le \frac{2\beta}{1-\max\eta}\|f(t,\cdot)\|^2_{L^1(\Omega)}, \tag{7}$$

$$\|Q(f,f)(t,\cdot)\|_{L^1(\Omega)} \le \left(\frac{2}{1-\max\eta}+1\right)\beta\ \|f(t,\cdot)\|^2_{L^1(\Omega)}. \tag{8}$$

Existence of a nonnegative solution to (3)–(4) can be deduced by construction, starting from the following result, again proven in [7].

Proposition A.3. *Consider the initial-boundary value problem for the unknown* $v=v(t,x)$, $x\in\Omega$ *and* $t\in[0,T]$,

$$v_t - [\alpha(x)v_x]_x + \mu v = g, \quad \mu\ge 0, \tag{9}$$

with initial condition

$$v(0,\cdot) = v^{\mathrm{in}} \tag{10}$$

and boundary conditions

$$\lim_{x\to\pm 1}\alpha(x)v_x(t,x) = 0 \qquad a.e.\ t>0, \tag{11}$$

where $v^{\mathrm{in}}\in L^1(\Omega)$, $g\in C([0,T];L^1(\Omega))$ *are nonnegative functions. Then* (9)–(11) *admits a unique solution* $v\in C^0([0,T];L^1(\Omega))$, *and* v *is nonnegative.*

The procedure to build a nonnegative weak solution is based on the monotonicity properties of our problem.

Theorem A.4. *Let* f^{in} *a nonnegative function in* $L^1(\Omega)$. *Then there exists a nonnegative weak solution* $f\in L^\infty(0,T;L^1(\Omega))$ *to* (3)–(4).

Proof. Set

$$\rho = \int_\Omega f^{\mathrm{in}}(x_*)\mathrm{d}x_*.$$

Consider the sequence $(f^n)_{n\in\mathbb{N}}$ of functions, defined as solutions to

$$\int_\Omega \partial_t f^{n+1}\,\varphi\,\mathrm{d}x - \int_\Omega \partial_x(\alpha\partial_x\varphi)\,f^{n+1}\,\mathrm{d}x \\ + \left[\beta\rho + \sum_j \theta_j(t)\right]\int_\Omega f^{n+1}\,\varphi\,\mathrm{d}x = \langle Q^+(f^n,f^n),\varphi\rangle + \sum_j \theta_j(t)\int_\Omega f^n\,\varphi(\bar{x}_j)\,\mathrm{d}x, \tag{12}$$

with $f^0\equiv 0$, for all $\varphi\in C^2(\bar\Omega)$, satisfying the initial and boundary conditions

$$f^n(0,\cdot) = f^{\mathrm{in}},\quad n\geq 1,\qquad \lim_{x\to\pm 1}\alpha(x)f^n_x(t,x) = 0 \text{ for a.e. } t>0,\ n\geq 1.$$

Thanks to Lemma A.2, we can apply Proposition A.3 and deduce, by induction, that f^n exists, belongs to $C^0([0,T];L^1(\Omega))$ and is nonnegative.

Then, choosing $\varphi = 1$ in (12), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega f^{n+1}\,\mathrm{d}x + \left[\beta\rho + \sum_j\theta_j(t)\right]\int_\Omega f^{n+1}\,\mathrm{d}x \\ = \beta\left(\int_\Omega f^n\,\mathrm{d}x\right)^2 + \sum_j\theta_j(t)\int_\Omega f^n\,\mathrm{d}x.$$

Therefore, by finite induction, we immediately get

$$\int_\Omega f^n\mathrm{d}x \leq \rho,\qquad n\geq 1.$$

Moreover, by applying the same strategy as in [7], we can prove that (f^n) is a non-decreasing sequence.

By monotone convergence, we finally deduce the existence of f as the limit of (f^n) in $L^\infty(0,T;L^1(\Omega))$. In order to check that f satisfies (3), we write

$$\int_0^T\int_\Omega f^{n+1}\ \varphi(x)\ \psi(t)\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_\Omega \big(\alpha(x)\ \varphi'(x)\big)'\ f^{n+1}\ \psi(t)\,\mathrm{d}x\,\mathrm{d}t \\ + \left(\beta\rho + \sum_j\theta_j(t)\right)\int_0^T\int_\Omega f^{n+1}\ \varphi(x)\ \psi(t)\,\mathrm{d}x\,\mathrm{d}t \\ = \beta\int_0^T\iint_{\Omega^2} f^n(t,x)f^n(t,x_*)\,\varphi(x')\,\psi(t)\,\mathrm{d}x\,\mathrm{d}x_*\,\mathrm{d}t \\ + \sum_j\int_0^T\int_\Omega \theta_j(t)\,f^n\,\varphi(\bar{x}_j)\ \psi(t)\,\mathrm{d}x\,\mathrm{d}t.$$

The only small difficulty lies in the fourth integral. It also converges since

$$\iint_{\Omega^2} |f^n(x)f^n(x_*) - f(x)f(x_*)|\ |\varphi(x')|\mathrm{d}x\mathrm{d}x_* \leq 2\rho\|f^n(t,\cdot)-f(t,\cdot)\|_{L^1}\|\varphi\|_{L^\infty}.$$

That ends the proof of Theorem A.4. □

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