# On the Optimal Choice of Coefficients in a Truncated Wild Sum and Approximate Solutions for the Kac Equation

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We study an approximate solution of the Boltzmann problem for Kac's caricature of a Maxwellian gas by using a truncated and modified expansion of Wild type. We choose the coefficients in the Wild sum approximation using a criterion based on exactly reproducing the behavior of the leading modes.

## 1. INTRODUCTION

In recent years much attention has been devoted to the numerical resolution of the Boltzmann equation (see, for example, refs. 6, 11, and 12). In this paper we deal with the Kac's caricature of a one dimensional Maxwell gas refs. 5 and 8. In Kac's model, the equation governing the evolution of the velocity density $f(v, t)$, $v \in \mathbf{R}$, is

$$\frac{\partial}{\partial t} f(v, t) = \int_{\mathbf{R}} \int_{-\pi}^{\pi} \rho(\theta) [f(v^*, t) \, f(w^*, t) - f(v, t) \, f(w, t)] \, dw \, d\theta, \quad (1)$$

where the post-collisional velocities $(v, w)$ are related to the initial velocities $(v^*, w^*)$ by:

$$v^* = v \cos \theta - w \sin \theta, \qquad w^* = v \sin \theta + w \cos \theta,$$

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and $\rho(\theta)$ is an *even* probability density on $[-\pi, \pi]$. A direct calculation shows that the model preserves both the total mass and the energy, but not the momentum.

In the rest of the section we suppose that the initial density $f(v, 0) = f_0(v)$ is a non-negative function such that:

$$\int_{\mathbf{R}} f_0(v)\, dv = 1 \qquad \text{and} \qquad \int_{\mathbf{R}} v^2 f_0(v)\, dv = 1. \tag{2}$$

It is a well known result that Boltzmann's $H$-functional

$$H(f) = \int_{\mathbf{R}} f(v, t) \log f(v, t)\, dv$$

is a monotone non-increasing function of $t$ if $f(v, t)$ is a solution of (1) with initial data $f_0$. Moreover $f$ tends to the Maxwellian equilibrium distribution:

$$M(v) = \frac{1}{\sqrt{2\pi}}\, e^{-v^2/2}$$

for $t \to \infty$, provided that $f_0$ satisfies (2).

The short-time behavior of solutions of (1) is also of interest. If a splitting method is used to solve the *spatially inhomogeneous* version of (1), then one needs good approximate solutions to (1) at the time $t = \Delta t$ to implement the collision step (see refs. 6 and 11 for more information). This paper is focused on the problem of obtaining accurate and readily computable approximations to solutions of (1) at a fixed time $t$, which can be thought of as representing the time step in a splitting scheme.

It is a remarkable fact that there is a constructive method[13] for solving the spatially homogeneous Boltzmann equation for a Maxwellian gas, and this same method applies to Kac's caricature as well. The solution of the initial value problem (1), with initial datum $f(v, 0) = f_0(v) \in L^1(\mathbf{R}) \cap L^\infty(\mathbf{R})$ can be expressed in terms of this initial data using a so-called Wild sum, as shown by Wild[13] in the case of a Maxwellian gas.

More explicitly, let us introduce the *Wild convolution* $f \circ g(v)$ of two probability densities $f$ and $g$ through

$$f \circ g(v) = \int_{\mathbf{R}} \int_{-\pi}^{\pi} \rho(\theta)\, f(v^*, t)\, g(w^*, t)\, dw\, d\theta.$$

Then the initial value problem for (1) can be written

$$\frac{\partial f}{\partial t} = f \circ f - f \qquad f(v, 0) = f_0(v),$$

and the Wild sum solution, given in this case by McKean,[10] is

$$f(v, t) = e^{-t} \sum_{n=0}^{\infty} (1 - e^{-t})^n f_n(v), \tag{3}$$

where

$$f_{n+1}(v) = \frac{1}{n+1} \sum_{k=0}^{n} f_k \circ f_{n-k}(v).$$

If we fix any positive integer $N$, we obtain an approximate solution of (1) by setting $f^{(N)}(v, t) = e^{-t} \sum_{n=0}^{N} (1 - e^{-t})^n f_n(v) + (1 - e^{-t})^{N+1} M(v)$, where $M(v)$ is the Maxwellian density specified above. Qualitatively sharp bounds on how $\|f - f^{(N)}\|_{L^1}$ decreases to zero with $N$ have been obtained in ref. 3.

A truncated Wild's expansion has been used in ref. 6 for constructing numerical schemes, by considering the first terms up to the order $N$ in (3), and then adding the Maxwellian, multiplied by a coefficient, such that to satisfy the mass conservation.

This idea has given rise to the derivation of robust numerical schemes, i.e., unconditionally stable methods preserving the asymptotics of the fluid dynamic limit, called *time relaxed schemes*, which have the property, shown in ref. 6, to be of order $N$ in time.

For numerical purposes, it is unlikely to be feasible to take $N > 1$ because of the expense of computing Wild convolutions. It has been observed by Pareschi and Russo[11] that, if one fixes $N = 1$, one can get better approximate solutions by using

$$f(v, t) \approx A_0(t) f_0(v) + A_1(t) f_0 \circ f_0(v) + A_2(t) M(v) \tag{4}$$

for other choices of $A_0(t)$, $A_1(t)$, and $A_2(t)$ than those corresponding to the approximation $f(v, t) \approx f^{(1)}(v, t)$.

In general, the problem as formulated by Pareschi and Russo is to choose the coefficients $A_k(t)$ in

$$f^{(N)}(v, t) = \sum_{k=0}^{N} A_k f_k(v) + A_{N+1}(t) M(v) \tag{5}$$

so that $f^{(N)}(v, t)$ gives a good approximation to the true solution at the fixed time $t$.

The weights $A_k(t)$ are non-negative functions satisfying the following properties:

(i)   consistency:

$$\lim_{t \to 0} A_1(t)/t = 1,$$

$$\lim_{t \to 0} A_k(t)/t = 0, \qquad k = 2,..., N+1;$$

(6)

(ii)  mass conservation:

$$\sum_{k=0}^{N+1} A_k(t) = 1, \qquad t \in [0, \infty);$$

(7)

(iii) asymptotic relaxation:

$$\lim_{t \to \infty} A_k(t) = 0, \qquad k = 0,..., N.$$

There are several systems of weights $A_k$ which satisfy properties (i), (ii), (iii). In particular, the choice

$$A_k = e^{-t}(1 - e^{-t})^k,$$
$$A_{k+1} = (1 - e^{-t})^{N+1}, \qquad k = 0,..., N$$

gives the Wild's coefficients.

While Pareschi and Russo have shown that there exist sets of weights $A_k$ that give better approximations for fixed small $N$ in numerical computations than do the Wild coefficients, they leave open the problem of how to optimally determine these weights. Here we give a solution to such a problem.

In this paper we limit our attention primarily to second-order schemes, leaving to the reader the straightforward generalizations.

The starting point is to write the initial data $f_0$ as

$$f_0(v) = M(v)(1 + h_0(v)) \tag{8}$$

where, on account of (2),

$$\int_{\mathbf{R}} h_0(v) \, M(v) \, dv = 0 \qquad \text{and} \qquad \int_{\mathbf{R}} v^2 h_0(v) \, M(v) \, dv = 0. \tag{9}$$

We then consider the spectral decomposition of $h_0(v)$ with respect to the linearized collision operator

$$\mathcal{L}h(v) = \frac{1}{M(v)}\left(M \circ (Mh)(v) + (Mh \circ M)(v)\right) - h(v). \tag{10}$$

As shown by McKean, the eigenfunctions of $\mathcal{L}$ are the Hermite polynomials $H_k(v)$, and the corresponding eigenvalues $\lambda_k$ are given by

$$\lambda_k = \int_{-\pi}^{\pi} \rho(\theta)(\cos^k(\theta) + \sin^k(\theta) - 1)\, d\theta. \tag{11}$$

Then we can write $h_0$ in terms of its spectral expansion:

$$h_0(v) = \sum_{k=1}^{\infty} a_k H_k(v). \tag{12}$$

Of course some modes in this expansion will be more important than others, depending on how large $a_k$ is, and when $t$ is not too small, on what $\lambda_k$ is. In particular, $a_2 = 0$ by the conservation of mass and energy. *Our approach is to choose the coefficients in such a way that the most relevant modes are treated exactly*. In general though, which modes matter most will depend on the type of problem under consideration.

For example, when spatially inhomogeneous kinetic equations are used to study hydrodynamic evolution, one typically starts from local Maxwellian initial data.

Even if the hydrodynamics of the inhomogeneous Kac's model is not clear, since the Gaussians are centered (and therefore the resulting macroscopic velocity is zero), it is nevertheless possible to consider data of the form

$$\rho(x)\,\frac{1}{\sqrt{2\pi T(x)}}\,e^{-v^2/2T(x)},$$

to study the approach to the equilibrium and find the leading modes.

One step of "streaming," that is collisionless flow $(x, v) \to (x + v\,\Delta t, v)$ changes the phase space density to

$$
\begin{aligned}
\rho(x - v\,\Delta t)&\,\frac{1}{\sqrt{2\pi T(x - v\,\Delta t)}}\,e^{-v^2/2T(x - v\,\Delta t)} \\
&= \frac{\rho(x)\,e^{-v^2/2T(x)}}{\sqrt{2\pi T(x)}}\left[1 + \left(\left(\ln\frac{T(x)^{1/2}}{\rho(x)}\right)' v - \frac{v^3}{2T(x)}\,(\ln T(x))'\right)\Delta t + \mathcal{O}(\Delta t^2)\right].
\end{aligned}
$$

Thus, in this type of problems, we expect the coefficients $a_1$ and $a_3$ in (12) to be of order $\Delta t$, while all others will be of order $\Delta t^2$ or smaller. Moreover, it is clear that this situation will persist in subsequent streaming and collision steps. This suggests that, in this case, particular attention should be paid to the $k = 1$ and $k = 3$ modes. As we shall see, this is correct.

Once we have decided which modes are most significant in a given problem, we proceed to determine the coefficients $A_0(t)$, $A_1(t)$ and $A_2(t)$ in (4) by a moment matching method.

To do this, we use the fact that we can compute exactly the full non-linear evolution of the moments

$$Q_k(f) = \int_{\mathbf{R}} v^k f(v) \, dv, \qquad k \in \mathbf{N}$$

using a method of Inkenberry and Truesdell.[7] We then require that

$$A_0(t) \, Q_k(f_0) + A_1(t) \, Q_k(f_0 \circ f_0) + A_2(t) \, Q_k(M) = Q_k(f(\cdot, t)) \quad (13)$$

for selected values of $k$. Though the right hand side is computed for the *exact non-linear evolution*, as we shall see, the eigenvalues of the linearized operator determine the behavior of the $Q_k(f(\cdot, t))$.

Since $A_0(t) + A_1(t) + A_2(t) = 1$, there are only two degrees of freedom, and we can in general only hope to have this hold for two values of $k$, but see Theorem 3.

*We shall show here that no matter which two choices are made for these values of $k$, the result is an explicit set of coefficients $A_0(t)$, $A_1(t)$ and $A_2(t)$ that is independent of $f_0$ and depends instead only on $t$ and $\rho(\theta)$.* This universal property of the optimal coefficients is very favorable for application since then the coefficients can be computed once and for all, in advance. It is what selects the moment matching method as particularly adapted to the problem at hand. If we had sought instead to choose $A_0(t)$, $A_1(t)$ and $A_2(t)$ so as to minimize, say,

$$\|f(\cdot, t) - (A_0(t) \, f_0 + A_1(t) \, f_0 \circ f_0 + A_2(t) \, M)\|_1,$$

this would not be the case and, at each step, the choice of the coefficients would depend on the solution of a difficult variational problem.

Not only is the universality of the coefficients very favorable for computation, these coefficients yield very accurate results for appropriate initial data. For example, in "hydrodynamic" problems of the type discussed above, it is natural to consider initial data $f_0$ of the form

$$f_0(v) = M(v)(1 + av + b(v^3 - 3v) + h(v)) \quad (14)$$

where $a$ and $b$ are of order $t$ and $h$ is a linear combination of Hermite polynomials of fourth degree and higher, and is such that $\|hM\|_1$ is of order $t^2$. Let $\tilde{f}_0$ be defined by

$$\tilde{f}_0(v) = M(v)(1 + av + b(v^3 - 3v)). \tag{15}$$

Notice that $\tilde{f}_0(v)$ is negative for some values of $v$, so it does not represent physical initial data. However, the evolution under (1) is stable so that the solution of (1) with initial data $\tilde{f}_0$ remains close to the solution of (1) with initial data $f_0$ on $[0, t]$. The second good point is that (1) is *exactly solvable* with the initial data $\tilde{f}_0$. In fact, a more general result is true, and is easily derived from the following theorem, special cases of which were noted in ref. 4.

**Theorem 1 (Exact Solutions of the Kac Equation).** Let $\rho(\theta)$ be even, and let $p(v)$ be any odd polynomial in $v$. Then

$$Mp \circ Mp = 0. \tag{16}$$

In particular, if $p(v, t)$ is an odd polynomial with coefficients depending on $t$ in such a way that $M(v)(1 + p(v, t))$ is a solution of the linearized Kac equation, then $M(v)(1 + p(v, t))$ is a solution of the full Kac equation.

*Proof.* It suffices to show that $Mv^m \circ Mv^n = 0$ when $n$ is odd. By direct computation,

$$Mv^m \circ Mv^n = \sum_{j=0}^{n} \sum_{i=0}^{m} \binom{n}{j}\binom{m}{i} M(v)\, v^{m-i+j}\, I_1(m, n, i, j)\, I_2(m, n, i, j)$$

where

$$I_1(m, n, i, j) = \int_{-\pi}^{\pi} \rho(\theta)\, \cos^{m+n-i-j}\theta\, \sin^{i+j}\theta\, d\theta$$

and

$$I_2(m, n, i, j) = \int_{\mathbf{R}} M(w)\, w^{n-j+i}\, dw.$$

But the Gaussian integral vanishes unless $n - j + i$ is even, and the trigonometric integral vanishes unless $i + j$ is even. If $n$ is odd, these cannot both be true, and therefore one factor or the other must vanish. The final statement now follows immediately from the first part.

For example, consider

$$f(v, t) = M(v)(1 + a(t)\, v + b(t)(v^3 - 3v)). \tag{17}$$

One easily computes that

$$f \circ f(v, t) - f(v, t) = M(v)(a(t)(c_1 - 1)\, v + b(t)(c_3 - 1)(v^3 - 3v))$$

where

$$c_k = \int_{-\pi}^{\pi} \rho(\theta) \cos^k \theta \, d\theta. \tag{18}$$

Therefore, (17) defines an exact solution of (1) in case

$$a(t) = a_0 e^{(c_1 - 1)\, t} \qquad \text{and} \qquad b(t) = b_0 e^{(c_3 - 1)\, t}.$$

We now show that matching the $k$ th moments as in (13) leads to exactly this solution.

**Theorem 2 (Optimal Coefficients Independent of $f_0$).** In order that

$$A_0(t)\, Q_k(f_0) + A_1(t)\, Q_k(f_0 \circ f_0) + A_2(t)\, Q_k(M) = Q_k(f(\,\cdot\,, t)) \tag{19}$$

holds for $k = 1$ and $k = 3$, one must have, independently of $f_0$,

$$A_0(t) = \frac{c_1 e^{(c_3 - 1)\, t} - c_3 e^{(c_1 - 1)\, t}}{c_1 - c_3} \qquad \text{and} \qquad A_1(t) = \frac{e^{(c_3 - 1)\, t} - e^{(c_1 - 1)\, t}}{c_3 - c_1} \tag{20}$$

where $c_k$ is given by (18). When the initial datum $f_0$ is given by (15), then with the coefficients $A_0(t)$, $A_1(t)$ and $A_2(t)$ determined by (20), $A_0(t)\, f_0 + A_1(t)\, f_0 \circ f_0 + A_2(t)\, M$ is the exact solution.

We prove this theorem in Section 2, after having first obtained formulae for the evolution of the moments. We will use these to prove an analogous of Theorem 2, where other values of $k$ are used in (13) to determine the coefficients. We also examine in more detail the circumstances in which (20) leads to highly accurate approximate solutions. Section 3 is devoted to a numerical investigation of some cases in which even values of $k$ play the dominant role. Finally, Section 4 contains some concluding remarks.

## 2. TIME EVOLUTION OF THE MOMENTS

Consider Eq. (1). Multiplying both sides of that equation by $v^k$ and integrating with respect to $v$ over $\mathbf{R}$, we obtain:

$$\frac{d}{dt} \int_{\mathbf{R}} v^k f(v, t) \, dv$$

$$= \int_{\mathbf{R} \times \mathbf{R}} \int_{-\pi}^{\pi} \rho(\theta) \, v^k [f(v^*, t) \, f(w^*, t) - f(v, t) \, f(w, t)] \, dv \, dw \, d\theta, \quad (21)$$

which can be written as

$$\frac{d}{dt} Q_k(f) = Q_k(f \circ f) - Q_k(f).$$

A straightforward computation shows that the Jacobian of the transformation $(v, w) \to (v^*, w^*)$ is equal to 1, and for all $k \in \mathbf{N}$ we have:

$$Q_k(f \circ f) = \int_{\mathbf{R} \times \mathbf{R}} \int_{-\pi}^{\pi} \rho(\theta) \, v^k f(v^*) \, f(w^*) \, dv \, dw \, d\theta$$

$$= \int_{\mathbf{R} \times \mathbf{R}} \int_{-\pi}^{\pi} \rho(\theta)(v \cos \theta + w \sin \theta)^k f(v) \, f(w) \, dv \, dw \, d\theta.$$

Therefore,

$$Q_k(f \circ f) = \sum_{i=0}^{k} \left[ Q_{k-i}(t) \, Q_i(t) \binom{k}{i} \int_{-\pi}^{\pi} \rho(\theta) \cos^{k-i} \theta \sin^i \theta \, d\theta \right]. \quad (22)$$

Since by hypothesis $\rho(\theta)$ is even,

$$\int_{-\pi}^{\pi} \rho(\theta) \cos^{k-i} \theta \sin^i \theta \, d\theta = 0$$

for all odd values of $i$. Hence,

$$Q_k(f \circ f) = \left( \int_{-\pi}^{\pi} \rho(\theta)(\cos^k \theta + \sin^k \theta) \, d\theta \right) Q_k(f)$$

$$+ \sum_{i=2, \, i \text{ even}}^{k-1} \left[ Q_{k-i}(t) \, Q_i(t) \binom{k}{i} \int_{-\pi}^{\pi} \rho(\theta) \cos^{k-i} \theta \sin^i \theta \, d\theta \right].$$

Therefore,

$$\frac{d}{dt}Q_k(f) = \lambda_k Q_k(f) + \sum_{i=2,\,i\,\text{even}}^{k-1} B_{k,i} Q_{k-i}(f)\, Q_i(f),$$

where

$$B_{k,i} = \binom{k}{i} \int_{-\pi}^{\pi} \rho(\theta) \cos^{k-i}\theta \sin^i\theta \, d\theta.$$

Notice that the $\lambda_k$ and the $B_{k,i}$ depend only on $\rho(\theta)$ and not on $f(v)$. The $Q_k(f)$ satisfy a system of differential equations whose coefficients are independent of the initial data $f_0$. Moreover, the coefficient of $Q_k(f)$ in its time derivative is $\lambda_k$, the corresponding eigenvalue of the linearized evolution.

Now let us apply this to find the optimal coefficients $A_0(t)$, $A_1(t)$ and $A_2(t)$ when we choose $k=1$ and $k=3$. Then we have from the above equations that

$$\frac{d}{dt}Q_1(f) = \lambda_1 Q_1(f) = (c_1 - 1)\, Q_1(f),$$

where $c_k$ is given by (18), so that

$$Q_1(f(\,\cdot\,,t)) = e^{(c_1-1)\,t} Q_1(f_0).$$

Similarly, the combination $Q_3(f) - 3Q_1(f)$ is found to satisfy

$$(Q_3(f(\,\cdot\,,t)) - 3Q_1(f(\,\cdot\,,t))) = e^{(c_3-1)\,t}(Q_3(f_0) - 3Q_1(f_0)).$$

This leads to the system of equations:

$$(A_0(t) + c_1 A_1(t) - e^{(c_1-1)\,t})\, Q_1(f_0) = 0$$

$$(A_0(t) + c_3 A_1(t) - e^{(c_3-1)\,t})(Q_3(f_0) - 3Q_1(f_0)) = 0.$$

Since the previous equations should be valid for every choice of the initial data, this means that

$$(A_0(t) + c_1 A_1(t) - e^{(c_1-1)\,t}) = 0$$

$$(A_0(t) + c_3 A_1(t) - e^{(c_3-1)\,t}) = 0.$$

Notice two things: when $k$ is odd, $A_2(t)$ will not appear since all odd moments of $M$ vanish, and this system does not involve $f_0$. It is easily solved to yield

$$A_0(t) = \frac{c_1 e^{(c_3-1)t} - c_3 e^{(c_1-1)t}}{c_1 - c_3}$$

and

$$A_1(t) = \frac{e^{(c_3-1)t} - e^{(c_1-1)t}}{c_3 - c_1}.$$

This proves Theorem 2.

For physical (non-negative) initial data $f_0$ of the type given in (14), the approximate solution given by (19) will in fact only be approximate. But because of the $L^1$ stability of the Kac equation,[1] if the difference between the physical data $f_0(v) = M(v)(1 + av + b(v^3 - 3v) + h(v))$ and $\tilde{f}_0(v) = M(v)(1 + av + b(v^3 - 3v))$ is of order $t^2$ in $L^1$, then the corresponding solutions will satisfy $\|f(t, \cdot) - \tilde{f}(t, \cdot)\|_1 = \mathcal{O}(t^2)$. But since $\tilde{f}(v, t)$ is given by (19) with the coefficients given by (20), we see that in this case, which is of special importance due to the connection with hydrodynamics, Theorem 2 provides particularly accurate approximations.

We now turn to the analogous of Theorem 2 in which we use other values of $k$ than $k = 1$ and $k = 3$. When we use even moments, the moments of $M$ will enter the system. Since the $k$th moment of the Maxwellian, for even $k \geqslant 0$, is given by

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} v^k e^{-v^2/2} dv = (k-1)!!,$$

for even $k$ we must solve

$$A_0(t) \, Q_k(f_0) + A_1(t) \int_{\mathbf{R}} v^k (f_0 \circ f_0) \, dv + A_2(t)(k-1)!! = Q_k(f(\cdot, t)). \quad (23)$$

By considering equation (22), it is possible to simplify this, obtaining the equation representing the conservation of any even moment:

$$
\begin{aligned}
A_0(t) \, &Q_k(f_0) \\
&+ A_1(t) \sum_{i=0}^{k} \left[ Q_{k-i}(f_0) \, Q_i(f_0) \binom{k}{i} \int_{-\pi}^{\pi} \rho(\theta) \cos^{k-i}\theta \, \sin^i\theta \, d\theta \right] \\
&+ A_2(t)(k-1)!! \\
&= Q_k(f(\cdot, t)).
\end{aligned}
\quad (24)
$$

A case in which it is required to use an even moment is considered in the next section.


## 3. THE ISOTROPIC KAC'S MODEL

In this section we show a situation in which the choice of the moments of order $k = 1$ and $k = 3$ is not allowed. This happens when $\rho(\theta)$ is constant, i.e., when we consider isotropic scattering with respect to $\theta$. Since $\rho(\theta)$ should be a probability density on $[-\pi, \pi]$, this means $\rho(\theta) = 1/2\pi$.

In this case, indeed, the odd moments of the Maxwellian in (13) vanish, and, by Eq. (22), also $Q_k(f \circ f) = 0$. This means that every odd moment is governed by the following linear ODE:

$$Q_k(f(t)) = Q_k(f_0)\, e^{-t}, \qquad k = 1, 3, 5, \dots \qquad (25)$$

and hence, that (13) reduces to $A_0(t)\, Q_k(f_0) = e^{-t} Q_k(f_0)$, which simply says $A_0(t) = e^{-t}$, independent of $k$. Thus, all odd moments impose the same requirement on the coefficients.

Therefore, to determine the coefficients, we will take $k = 1$ together with an even moment. The inclusion of $k = 1$ means that we exactly reproduce the behavior of all of the infinitely many odd moments. As for the even moment, the fact that the 4th degree Hermite polynomial is the mode of the linearized Kac equation, for $\rho$ constant, with the slowest relaxation argues for choosing $k = 4$. An analogous of Theorem 2 can now be stated:


**Theorem 3 (Optimal Coefficients for the Uniform Kac Equation).**
In order that

$$A_0(t)\, Q_k(f_0) + A_1(t)\, Q_k(f_0 \circ f_0) + A_2(t)\, Q_k(M) = Q_k(f(\cdot, t)) \qquad (26)$$

holds for $k = 1$ and $k = 4$, one must have, independent of $f_0$,

$$A_0(t) = e^{-t}, \qquad A_1(t) = \tfrac{4}{3}(e^{-t/4} - e^{-t}), \qquad A_2(t) = 1 + \tfrac{1}{3}(e^{-t} - 4e^{-t/4}). \qquad (27)$$

*Proof.* By the conservation of the mass we have $A_0(t) + A_1(t) + A_2(t) = 1$. Then we apply Eqs. (25) to (13) to obtain $A_0(t) = e^{-t}$. Finally, since the solution of the equation governing the fourth moment:

$$\frac{d}{dt} Q_4(f) = -\frac{1}{4} Q_4(f) + \frac{3}{4}$$

has the solution

$$Q_4(f) = [Q_4(f_0) - 3] e^{-t/4} + 3,$$

by inserting this result in (13) we obtain the thesis.

We point out that, as distinct from the situation explored in the previous section, here the resulting scheme is only first-order accurate, and this is in agreement with the results of Gabetta, Pareschi and Toscani.[6]

On the other hand, the coefficients of the Wild-like expansion obtained in this way give a better accuracy than the Wild coefficients, as we will see in the numerical examples that follow. In all tests of this section we compare the approximation obtained with the method described in the paper with the following exact solution of the Kac equation:[2,9]

$$f_E(v, t) = \tfrac{1}{2}[\tfrac{3}{2}(1 - C(t))\sqrt{C(t)} + (3C(t) - 1) C(t)^{3/2} v^2] e^{-C(t) v^2},$$

where

$$C(t) = \frac{1}{3 - 2e^{-\sqrt{\pi} t/16}}.$$

Note that

$$\int_{\mathbf{R}} f_E(v, t) \, dv = \frac{\sqrt{\pi}}{2} \qquad \text{and} \qquad \int_{\mathbf{R}} v^2 f_E(v, t) \, dv = \frac{3}{4}\sqrt{\pi},$$

and therefore some of the formulae appearing in the paper have been here adapted to such a situation.

Let's denote by $E_M(t)$ the $L^1$-norm of the difference between the exact solution and the approximation provided by Theorem 3; i.e.,

$$E_M(t) = \|f(\cdot, t) - (A_0(t) f_0 + A_1(t) f_0 \circ f_0 + A_2(t) M)\|_1,$$

where $A_0(t)$, $A_1(t)$ and $A_2(t)$ are given as in Theorem 3.

In Figs. 1 and 2 we show the behavior of $E_M(t)$ at various time steps. It is clear from Fig. 1 that the approximation possesses the correct asymptotics, while in intermediate times gives a less accurate result.

Figure 3 is devoted to the comparison, for $t = 1$, between the exact solution $f_E(v, t = 1)$ and its approximation of order 1 with the method of moments

$$f_E^{(1)}(v, t = 1) = A_0(1) f_0 + A_1(1) f_0 \circ f_0 + A_2(1) M,$$

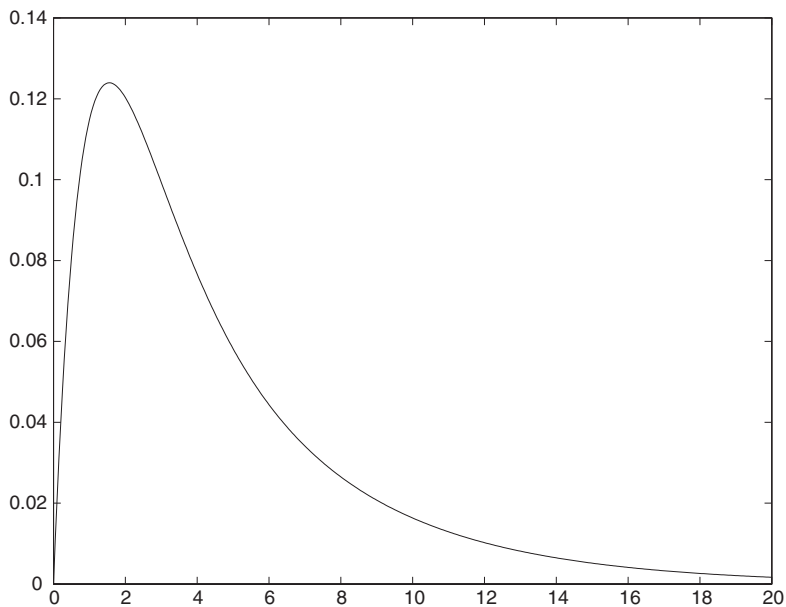that is we have chosen a time instant in the region of bad accuracy.

Fig. 1.    Behavior of $E_M(t)$ with respect to the time for $0 \leqslant t \leqslant 20$.
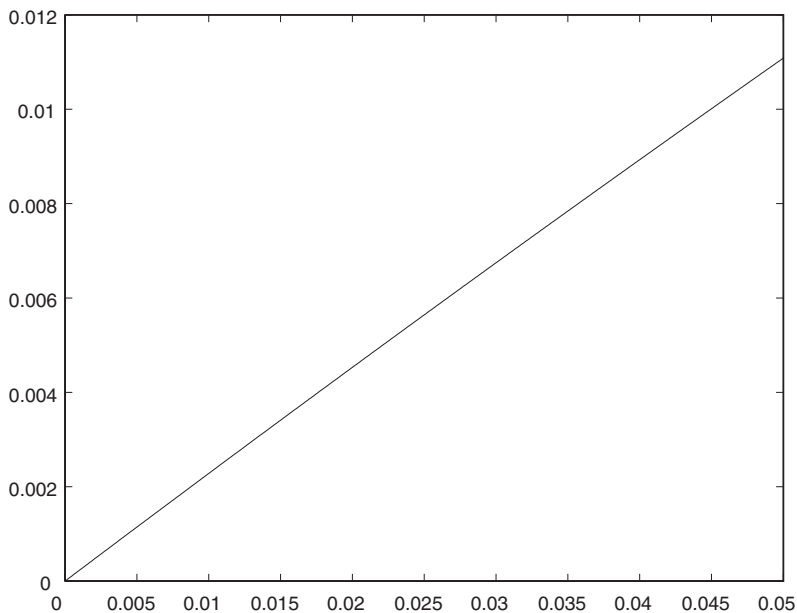


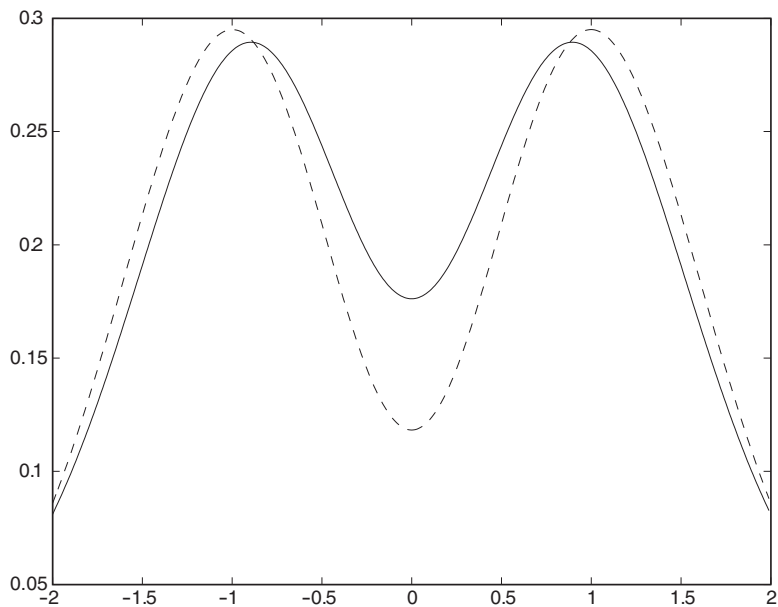Fig. 2.    Detail of Fig. 1 for small times.

Fig. 3.   Comparison between the exact solution $f_E$ (dashed line) and its approximation $f_E^{(1)}$ (continuous line) with the method of moments for $t = 1$.
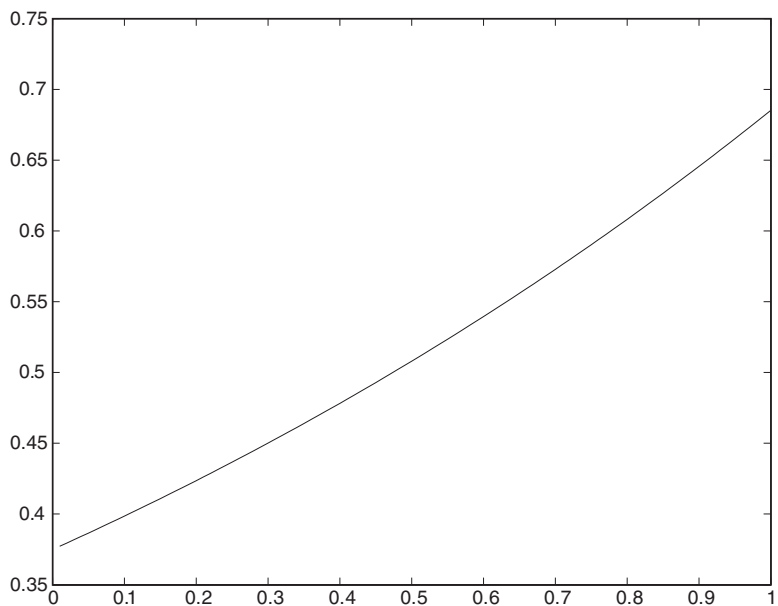


Fig. 4.   Behavior of the ratio $E_M(t)/E_W(t)$ with respect to the time for $0 < t \leqslant 1$.

Finally, in Fig. 4 we show the behavior of the ratio between $E_M(t)$ and $E_W(t)$ for $t \leqslant 1$.

The quantity $E_W(t)$ is the $L^1$-norm of the difference between the exact solution and the corresponding Wild's expansion up to the order 1, and it is given by

$$E_W(t) = \|f(\cdot, t) - (A_0(t) f_0 + A_1(t) f_0 \circ f_0 + A_2(t) M)\|_1,$$

where $A_0(t)$, $A_1(t)$ and $A_2(t)$ are the Wild's coefficients.

Since the ratio $E_M(t)/E_W(t) < 1$, we note that the accuracy of the methods of moments, in this case, is better than the Wild's sum. The improvement of accuracy depends on the time, and is more apparent for small times. Since the interest in practical applications is mainly in the behavior for small time steps, this feature is another advantage of the method when compared with a truncated Wild's sum.

## 4. CONCLUDING REMARKS

We have seen that moment matching criteria lead to the selection of coefficients in a truncated Wild sum that under certain physically interesting circumstances provide second order accuracy at first order cost. Moreover, the coefficients are universal, independent of the initial data, and can be computed once and for all in advance. This would not have been the case had we used, say, optimal fit in the $L^1$-norm to choose the coefficients. We finally remark that similar results will hold for the full Boltzmann equation for Maxwellian molecules.

## ACKNOWLEDGMENTS

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