ON THE EXPONENTIAL DECAY TO EQUILIBRIUM OF THE DEGENERATE LINEAR BOLTZMANN EQUATION

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Abstract. In this paper we study the decay to the equilibrium state for the solution of the linear Boltzmann equation in the torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, $d \in \mathbb{N}$, by allowing that the non-negative cross section σ can vanish in a subregion $X := \{x \in \mathbb{T}^d \,|\, \sigma(x) = 0\}$ of the domain with $\mathrm{meas}(X) \geq 0$ with respect to the Lebesgue measure.

We show that the geometrical characterization of X is the key property to produce exponential decay to equilibrium.

1. Introduction

The theory concerning the time asymptotics of the linear Boltzmann equation is largely known when the cross sections are bounded from below by a strictly positive constant. In this case, it is known that the solution converges exponentially fast to its equilibrium for a wide class of initial data and boundary conditions (see, for example, [11]).

The deep reason of this behaviour is the *local coercivity* of the scattering and absorption operators, which means that the solution is locally "attracted" everywhere toward its average. However, the situation is noticeably different in the degenerate case, i.e. when the cross sections vanish on some set where the particles move freely without changing of direction, and consequently where the solution is not "attracted" towards its average.

Actually, the characterization of the time asymptotics in the degenerate case is still an open problem.

A first partial answer to this question has been obtained by Desvillettes and Salvarani in a special situation, namely when the cross section vanishes at a finite number of points [5].

In their paper, they also conjectured that some explicit rate should still exist (though non necessarily exponential) even in degenerate situations, at least when the equilibrium is unique.

This conjecture is based on the intuition that the hypocoercivity properties of the degenerate linear Boltzmann equation are not given by the scattering and absorption terms alone (a property which is true only when the cross section is strictly bounded from below by a positive constant). They are rather obtained through the simultaneous action of the aforementioned operators together with the free transport operator.

We notice that, in general, the decay cannot be exponential, as shown by a counterexample of the authors in [3]. Indeed, we considered the linear Boltzmann equation on the torus $\mathbb{T}^d$, $d \geq 2$, with velocities on the sphere $\mathbb{S}^{d-1}$ and gave an example of initial condition $f^{\mathrm{in}} \in L^\infty(\mathbb{T}^d \times \mathbb{S}^{d-1})$ such that, for a wide class of cross sections in L^∞, the L^2-distance to equilibrium cannot decay faster than $t^{-1/2}$.

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The obstruction to the exponential convergence comes from the existence of infinite open channels where the particles move freely, which implies the impossibility of giving a finite uniform control of the exit time of the particles starting from inside such an open channel.

However, it is reasonable to conjecture that the degeneracy of the cross-section is not in itself an absolute obstacle to exponential decay to equilibrium.

In this paper we characterize the conditions that allow the uniform exponential time decay to the equilibrium state for the solution of the degenerate linear Boltzmann equation posed in the flat d-dimensional torus.

More precisely, we show that the exponential convergence of the solution to its asymptotic profile can be obtained, even in the degenerate case, under the hypothesis that the cross section satisfies a geometrical property, which we call the *geometrical condition* (see Section 2 below).

This result, which gives a necessary and sufficient condition for uniform exponential decay to equilibrium, agrees with the considerations of [3], where it was pointed out that the geometrical properties of the scattering region are the key feature of the problem.

2. The problem and the main result

Let $0 < v_m < v_M$ and consider $V = \{v \in \mathbb{R}^d \,:\, v_m \le |v| \le v_M\}$ with $d \ge 2$.

In this article, we study the asymptotic behaviour of the linear Boltzmann equation

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \sigma\,(f - Kf) = 0, & (t,x,v) \in \mathbb{R}_+ \times \mathbb{T}^d \times V \\ f(0,x,v) = f^{\mathrm{in}}(x,v) & (x,v) \in \mathbb{T}^d \times V, \end{cases} \tag{2.1}$$

where

$$Kf := \int_V k(v,w) f(t,x,w)\,dw. \tag{2.2}$$

The non-negative kernel k of the operator K is a function of class $L^\infty(V \times V)$ such that

$$\int_V k(v,w)\,dw = 1 \text{ and } k(v,w) > 0 \text{ a.e. on } V \times V. \tag{2.3}$$

We suppose moreover that

$$f^{\mathrm{in}} \in L^1\left(\mathbb{T}^d \times V\right)$$

and that

$$\sigma \in L^\infty\left(\mathbb{T}^d\right), \text{ with } \sigma \ge 0 \text{ a.e. and } \int_{\mathbb{T}^d} \sigma(x)dx > 0.$$

The measures on $\mathbb{T}^d \times V$ are normalized such that

$$\int_{\mathbb{T}^d} dx = \int_V dv = 1.$$

The function $f \equiv f(t,x,v)$ can be thought of as the density of particles that, at time $t > 0$, are located at $x \in \mathbb{T}^d$, with velocity $v \in V$. These particles do not interact between themselves, but only with the medium. The interaction with the background is described by the cross section σ, a function which represents the interaction rate between the particles and the medium at $x \in \mathbb{T}^d$, and by the kernel k.

The properties satisfied by σ that lead to an exponential convergence to equilibrium are summarized in the following definition.

Definition 2.1. *The cross section $\sigma \equiv \sigma(x)$ is said to verify the geometrical condition if there exist T_0 and $C > 0$ such that*

$$\int_0^{T_0} \sigma\left(\phi_{x,v}(s)\right) ds \geq C \text{ a.e. in } (x,v) \in \mathbb{T}^d \times V, \tag{2.4}$$

where $\phi_{x,v}$ designates the linear flow starting at $x \in \mathbb{T}^d$ in the direction $-v \in V$:

$$\phi_{x,v} : t \mapsto x - tv.$$

Notice that the geometrical condition entails that, for a.e. $(x,v) \in \mathbb{T}^d \times V$, there exists $t \in (0, T_0)$ such that $\phi_{x,v}(t) \in \left\{x \in \mathbb{T}^d \mid \sigma(x) > 0\right\}$. This property is reminiscent of the Bardos-Lebeau-Rauch condition[1] that guarantees the exponential stabilization of the wave equation [2].

We remark that, in one space dimension, the geometrical condition is always fulfilled for cross sections that are strictly positive on a sub-domain of the interval $(0,1)$ with positive Lebesgue measure, since $|v| \geq v_m > 0$.

Our main result is summarized in the following theorem:

Theorem 2.2. *Let $\sigma \in L^\infty\left(\mathbb{T}^d\right)$ be a non-negative cross section satisfying the geometrical condition* (2.4)*. Then there exist two constants $M > 0$ and $\alpha > 0$ such that the solution f of the Cauchy problem* (2.1) *satisfies the inequality*

$$\left\| f - \int_{\mathbb{T}^d \times V} f^{\text{in}}(x,v)\, dxdv \right\|_{L^1(\mathbb{T}^d \times V)} \leq M e^{-\alpha t} \left\| f^{\text{in}} \right\|_{L^1(\mathbb{T}^d \times V)} \tag{2.5}$$

for all $t \in \mathbb{R}_+$. Conversely, if the solution of the linear Boltzmann equation (2.1) *converges uniformly in L^1 to its equilibrium state at an exponential rate (i.e. satisfies* (2.5)*), then σ must satisfy the geometrical condition* (2.4).

3. The semigroup formulation of the problem

Our main arguments rely on the theory of positive strongly continuous semigroups in a Banach lattice. Before recalling some notions related to that theory, we restate the solution of the Cauchy problem (2.1) and the main theorem 2.2 in the frame of the semigroup theory.

Define the transport operator

$$B := A_0 - M_\sigma + K_\sigma \tag{3.1}$$

with domain

$$D(B) = \left\{ f \in L^1\left(\mathbb{T}^d \times\ V\right) \,\middle|\, v \cdot \nabla_x f \in L^1\left(\mathbb{T}^d \times\ V\right) \right\},$$

where A_0, M_σ and K_σ designate respectively the collisionless transport operator, the absorption operator and the scattering operator.

More precisely, the collisionless transport operator is

$$(A_0 f)(x,v) := -v \cdot \nabla_x f \text{ for each } f \in D(A_0),$$

with domain

$$D(A_0) = D(B);$$

the absorption and the scattering operator are defined respectively by

$$(M_\sigma f)(x,v) := \sigma(x) f(x,v) \text{ for each } f \in L^1\left(\mathbb{T}^d \times\ V\right)$$

and

$$(K_\sigma f)(x,v) := \sigma(x) \int_V k(v,w) f(x,w) dw \text{ for each } f \in L^1\left(\mathbb{T}^d \times\ V\right)$$

[1]The term "geometrical condition" was used in [2].

with

$$D(M_\sigma) = D(K_\sigma) = L^1\left(\mathbb{T}^d \times \ V\right).$$

The system (2.1) corresponds then to the abstract Cauchy problem

$$\begin{cases} \dfrac{d}{dt} f = Bf \\ \\ f(0,x,v) = f^{\text{in}}(x,v) \in \mathbb{T}^d \times \ V. \end{cases}$$

The operator B generates a strongly continuous semigroup $\mathcal{T} \equiv (T_t)_{t\geq 0}$ (see [6] for a proof). In other words, for each $f^{\text{in}} \in L^1\left(\mathbb{T}^d \times \ V\right)$, each mild solution f of the Cauchy problem (2.1) can be written in $C\left(\mathbb{R}_+; L^1\left(\mathbb{T}^d \times \ V\right)\right)$ as

$$f = T_t\left(f^{\text{in}}\right), \text{ where } T_t \in \mathcal{L}(L^1(\mathbb{T}^d \times V)).$$

Theorem 2.2 is equivalent to the existence of a pair (M, α) of positive constants such that

$$\|T_t - P\|_{\mathcal{L}(L^1(\mathbb{T}^d \times \ V))}(t) \leq M e^{-\alpha t},$$

where

$$P(f) = \int_{\mathbb{T}^d \times \ V} f(x,v)\,dx\,dv \text{ for each } f \in L^1\left(\mathbb{T}^d \times \ V\right).$$

Now we briefly recall some elements of the theory of Banach lattices and of positive semigroups used in our analysis.

The space $L^1\left(\mathbb{T}^d \times \ V\right)$ is an ordered set with the partial order defined by

$$f \geq 0 \text{ if and only if } f(x,v) \geq 0 \text{ a.e. on } \mathbb{T}^d \times V.$$

This space, endowed with the standard L^1-norm, is a Banach lattice, i.e. a real Banach space endowed with an ordering $\geq$ compatible with the vector structure such that, if $f, g \in L^1(\mathbb{T}^d \times \ V)$ and $|f| \geq |g|$, then $\|f\|_1 \geq \|g\|_1$.

The semigroup $\mathcal{T}$ preserves the positivity, namely

$$f \geq 0 \implies T_t f \geq 0 \text{ for each } t \geq 0.$$

Therefore, we can apply here the theory of strongly continuous positive semigroups in Banach lattices (see for instance [1]).

Remark 3.1. *Let E be a Banach lattice. The space $\mathcal{L}(E)$ of bounded operators on E can be ordered in the following way. Let $A, B \in \mathcal{L}(E)$ then*

$$0 \leq A \leq B \textit{ if and only if, for each nonnegative } x \in E, \ 0 \leq Ax \leq Bx.$$

In order to make the paper self-consistent, we recall some notions related to this theory and that will be useful in the next sections.

Definition 3.2. *A semigroup $\mathcal{G} \equiv (G_t)_{t\geq 0}$ is said to be quasi-compact on $L^1\left(\mathbb{T}^d \times V\right)$ if and only if there exist a compact operator C on $L^1\left(\mathbb{T}^d \times V\right)$ and a constant $t_0 > 0$ such that*

$$\|G_{t_0} - C\|_{\mathcal{L}(L^1(\mathbb{T}^d \times V))} < 1.$$

A closed vector subspace V of a Banach lattice E is called *order ideal* if, when $x \in V$ and $y \in E$, $|y| \leq |x|$ implies $y \in V$. We denote $\mathcal{I}(E)$ the subset of the order ideals of E.

Let G be a operator in a Banach lattice E. An order ideal V is a G-*invariant* if $G(V) \subset V$ (see definition 8.1 p. 186 in [9]). We denote the set of G-invariants by

$$\mathcal{I}(G) := \{V \in \mathcal{I}(E) \mid G(V) \subset V\}.$$

Obviously, for each bounded operator G on E, $\{0\}$ and $E \in \mathcal{I}(G)$.

Let $\mathcal{G} \equiv (G_t)_{t\geq 0}$ be a semigroup. We denote

$$\mathcal{I}(\mathcal{G}) := \bigcap_{t\geq 0} \mathcal{I}(G_t)$$

and we say that an order ideal V is a *$\mathcal{G}$-invariant* if $V \in \mathcal{I}(\mathcal{G})$.

Definition 3.3. *An operator $G \in \mathcal{L}\left(L^1\left(\mathbb{T}^d \times\ V\right)\right)$ is said to be irreducible if and only if*

$$\mathcal{I}(G) = \left\{\{0\}, L^1\left(\mathbb{T}^d \times\ V\right)\right\}.$$

Likewise, a semigroup $\mathcal{G}$ is irreducible if

$$\mathcal{I}(\mathcal{G}) = \left\{\{0\}, L^1\left(\mathbb{T}^d \times\ V\right)\right\}.$$

Remark 3.4. *Let $\mathcal{B}(X)$ denote the family of Borel sets on X. For each $(\Omega, J) \in \mathcal{B}\left(\mathbb{T}^d\right) \times \mathcal{B}(V)$, define the subspace*

$$I(\Omega, J) := \left\{f \in L^1\left(\mathbb{T}^d \times\ V\right) \mid \operatorname{supp} f \subseteq \Omega \times J\right\}$$

and denote

$$\mathcal{I} := \left\{I(\Omega, J) \;\middle|\; (\Omega, J) \in \mathcal{B}\left(\mathbb{T}^d\right) \times \mathcal{B}(V)\right\}.$$

Then we can identify the set of order ideals of $L^1\left(\mathbb{T}^d \times\ V\right)$ with $\mathcal{I}$ (see p. 158 in [9]*).*

The notion of spectral bound of an operator (see Definition 1.12 p. 57, Chapter II in [6]) plays a key role in our discussion.

Definition 3.5. *Let A be a (possibly unbounded) linear operator on a Banach space E. Its spectral bound $s(A)$ is defined as*

$$s(A) := \sup\left\{Re\ \lambda \,|\, \lambda \in \sigma(A)\right\}.$$

Let $\mathcal{G}$ be a C_0-semigroup on E with generator A. The spectral bound of $\mathcal{G}$ is defined as follows:

$$s(\mathcal{G}) := s(A).$$

With the notions recalled above, we can now state the following theorem, which will be the key argument in our proof to obtain the sufficiency of the geometrical condition (see Theorem 2.1, p. 343 in [1]):

Theorem 3.6. *Let $(G_t)_{t\geq 0}$ be a bounded, quasi-compact, irreducible, positive C_0-semigroup on $L^1\left(\mathbb{T}^d \times\ V\right)$ with spectral bound zero. Then there exist a positive rank-one projection P and suitable constants $C \geq 1$ and $a > 0$ such that*

$$\|G_t - P\|_{\mathcal{L}(L^1(\mathbb{T}^d \times\ V))} \leq Ce^{-at} \textit{ for each } t \geq 0.$$

Therefore, in order to prove the first statement in Theorem 2.2, it suffices to check, under the assumptions above, that

(1) the spectral bound of B is zero,
(2) $\mathcal{T}$ is irreducible, and that
(3) the geometrical condition (2.4) implies that $\mathcal{T}$ is quasi-compact.

The aforementioned properties of the operator B and of the semigroup $\mathcal{T}$ generated by B will be proved in sections 4 to 6. In section 7 we characterize the projection operator P. The necessity of the geometrical condition (2.4) for exponential convergence to equilibrium (i.e the converse part of Theorem 2.2) will be considered in section 8.

4. The spectral bound of $\mathcal{T}$

The first part of our argument consists in establishing that the spectral bound of the semigroup $\mathcal{T}$ generated by B is zero. The following result holds:

Proposition 4.1. *Let B be the transport operator defined in* (3.1) *with domain*

$$D(B) = \left\{ f \in L^1\left(\mathbb{T}^d \times\ V\right) \,\middle|\, v \cdot \nabla_x f \in L^1\left(\mathbb{T}^d \times\ V\right) \right\},$$

and let $\mathcal{T}$ be the semigroup generated by B. Then $s(\mathcal{T}) = s(B) = 0$.

Proof. Since $\mathcal{T}$ is a strongly continuous positive semigroup in $L^1\left(\mathbb{T}^d \times\ V\right)$, its spectral bound is, by Theorem 1.15 p. 358 in Chapter VI of [6], equal to its growth bound, which means that

$$s(B) = \omega_0(\mathcal{T}) := \inf\left\{ \omega \in \mathbb{R} \,\middle|\, \exists\ M \geq 1 : \|T_t\|_{\mathcal{L}(L^1(\mathbb{T}^d \times\ V))} \leq M e^{\omega t}\ \forall t \geq 0 \right\}.$$

Therefore, it is enough to prove that $\omega_0(\mathcal{T}) = 0$. But, by Proposition 2.2 p. 251 Chapter IV in [6], we have that

$$\omega_0(\mathcal{T}) = \frac{1}{t} \ln r(T_t) \ \text{ for each } t > 0, \tag{4.1}$$

where r is the spectral radius of the operator (i.e., for any linear operator A, $r(A) = \sup\{|\lambda| \,:\, \lambda \in \sigma(A)\}$).

We know that

$$r(T_t) \leq \|T_t\|_{\mathcal{L}(L^1(\mathbb{T}^d \times\ V))} = 1 \tag{4.2}$$

and that, for each $t \geq 0$,

$$T_t\left(\mathbb{1}_{\mathbb{T}^d \times\ V}\right) = \mathbb{1}_{\mathbb{T}^d \times\ V}.$$

In other words, for each $t \geq 0$, 1 is an eigenvalue for T_t. Hence, by (4.2),

$$r(T_t) = 1 \text{ for each } t \geq 0.$$

This property, together with equality (4.1), implies that

$$\omega_0(\mathcal{T}) = 0,$$

which is the desired conclusion. □

5. Irreducibility of $\mathcal{T}$

Having proved that the spectral bound of the strongly continuous semigroup $\mathcal{T}$ is zero, the next step consists in deducing the irreducibility of $\mathcal{T}$ in $L^1\left(\mathbb{T}^d \times V\right)$.

The key point of the proof is the following proposition, which is a particular case of Proposition 3.3 p. 307 in [1], written here in a notation that will help us identifying the operators pertaining to our problem:

Proposition 5.1. *Suppose that A_0 is the generator of a positive semigroup $\mathcal{S}$, further assume that K_σ is a bounded positive operator and M_σ is a bounded real multiplier. Let $\mathcal{T}$ be the semigroup generated by $B := A_0 + K_\sigma + M_\sigma$. For a closed order-ideal $I \subset E$ the following assertions are equivalent:*

- *I is $\mathcal{T}$-invariant.*
- *I is invariant both under K_σ and $\mathcal{S}$.*

Let A_t be the semigroup generated by A_0. Proposition 5.1 and equality (3.1) show that the irreducibility of $\mathcal{T}$ is implied by

$$\mathcal{I}(K_\sigma) \cap \mathcal{I}(A_t) = \left\{ \{0\}, L^1\left(\mathbb{T}^d \times\ V\right) \right\}.$$

This result is consequence of the following lemma:

Lemma 5.2. *Under the assumptions and with the notations above, we have*

(1) $\mathcal{I}(A_t) = \left\{ I\left(\mathbb{T}^d, J\right) \mid J \subseteq \mathcal{B}(V) \right\}$,

(2) $\mathcal{I}(K_\sigma) = \{ I(\Omega, V) \mid \Omega \subseteq \mathcal{B}(\mathbb{T}^d) \}$.

Proof.

(1) Each ideal in $L^1(\mathbb{T}^d \times V)$ is of the form $L^1(M)$, where $M \in \mathcal{B}(\mathbb{T}^d \times V)$. If $\text{meas}(M^c) > 0$, that implies that

$$\int_{\mathbb{T}^d \times V} f(x,v) g(x,v)\, dxdv = 0$$

for all $g \in L^\infty(M^c)$ such that $g > 0$. In other words, in order to prove the first item of the lemma, it suffices to show that, for all $M \in \mathcal{B}(\mathbb{T}^d)$ there exists $f \in I(A_t)$ such that

$$\int_{\mathbb{T}^d \times V} f(x,v) g(x,v)\, dxdv > 0,$$

for a nonnegative $g \in L^\infty(M^c)$.

Let $I(\Omega, J) \in \mathcal{I}(A_t)$, and let O be an open set included in Ω. We recall that the linear flow $\phi_{x,v} : t \mapsto x - tv$ is ergodic in $\mathbb{T}^d$ whenever $\{v_1, \dots, v_d\}$ is $\mathbb{Q}$-linearly independent in $\mathbb{R}$. In other words, the flow $\phi_{x,v}$ is almost surely transitive on $\mathbb{T}^d$. That means that for each open O' there exists $t_o > 0$ such that

$$\int_{\mathbb{T}^d \times V} (A_t \mathbb{1}_{O \times J})(x,v) \mathbb{1}_{O' \times J}(x)(v) dxdv > 0.$$

Therefore

$$\Omega = \mathbb{T}^d$$

and

$$\mathcal{I}(A_t) = \{ I(\mathbb{T}^d, J) \mid J \subseteq \mathcal{B}(V) \}.$$

(2) First recall that for each $f \in I(\Omega, J)$ we have

$$\begin{aligned}(K_\sigma f)(x,v) &= \sigma(x) \int_V k(v,w) f(x,w) dw \\ &= (M_\sigma K f)(x,v)\end{aligned}$$

where M_σ designates the multiplication (by σ) operator in $L^1(\mathbb{T}^d)$ and K the scattering operator in $L^1(V)$ defined in (2.2), so that

$$\mathcal{I}(K_\sigma) = \mathcal{I}(M_\sigma) \times \mathcal{I}(K).$$

This implies the desired conclusion by the irreducibility of K on $L^1(V)$.

□

The result of our analysis is summarized in the next proposition.

Proposition 5.3. *The semigroup $\mathcal{T}$ generated by the transport operator B is irreducible in $L^1(\mathbb{T}^d \times V)$.*

6. Quasi-compactness of $\mathcal{T}$

The last property needed in order to apply Theorem 3.6 is the quasi-compactness of $\mathcal{T}$ in $L^1(\mathbb{T}^d \times V)$.

6.1. **The essential spectrum.** We recall the notion of essential spectrum of an operator (see [6] p. 248).

Definition 6.1. *An operator $A \in \mathcal{L}(E)$ is a Fredholm operator if and only if*

$$\dim \operatorname{Ker} A < \infty \text{ and } \dim E/\mathrm{R}(A) < \infty.$$

The essential resolvent of A is

$$\rho_{\mathrm{ess}}(A) := \{\lambda \in \mathbb{C} \,|\, \lambda I - A \ \text{ is Fredholm}\},$$

and its essential spectrum is

$$\sigma_{\mathrm{ess}}(A) := \mathbb{C} \setminus \rho_{\mathrm{ess}}(A).$$

The essential radius of A is

$$r_{\mathrm{ess}}(A) := \sup\{|\lambda| \mid \lambda \in \sigma_{\mathrm{ess}}(A)\}.$$

We recall that the semigroup $\mathcal{T}$ is quasi-compact on $L^1(\mathbb{T}^d \times V)$ if and only if

$$\text{there exists } t_o > 0 \text{ such that } r_{\mathrm{ess}}(T_{t_o}) < 1. \tag{6.1}$$

For a proof of this result, see Proposition 3.5 p. 332 in [6].

6.2. **A control of the essential radius of $\mathcal{T}$.** First, notice that, defining $\mathcal{S} \equiv (S_t)_{t \geq 0}$ by the formula

$$S_t g(x,v) := e^{-\int_0^t \sigma(x-vs)ds} g(x-vt, v) \ \text{ for all } g \in L^1(\mathbb{T}^d \times V), \tag{6.2}$$

the semigroup $\mathcal{T}$ can be seen as a perturbation of $\mathcal{S}$ by Duhamel's formula

$$T_t = S_t + \int_0^t S_s K_\sigma T_{t-s} ds. \tag{6.3}$$

Hence we surmise that, under suitable assumptions on σ, the essential radius of T_t is controlled by the spectrum of S_t. More precisely, the purpose of the present subsection is to establish the following proposition:

Proposition 6.2. *Under the assumptions above we have, for each $t > 0$,*

$$r_{\mathrm{ess}}(T_t) \leq r(S_t).$$

First, we observe that (6.3) implies, for each $n \in \mathbb{N}^*$, that

$$T_t = \sum_{j=0}^{n-1} S_j(t) + R_n(t)$$

where

$$S_0(t) = S_t \ \forall t > 0,$$

and

$$S_k(t) := \int_0^t S_{t-s} K_\sigma S_{k-1}(s) ds, \ \forall t > 0, \ \forall k \geq 1,$$

while $R_n(t)$ is the remainder term

$$R_n(t) = \int_{\substack{s_1 + \cdots + s_n \leq t; \\ 0 \leq s_i}} S_{s_1} K_\sigma \dots K_\sigma S_{s_n} K_\sigma T_{t-s_1-\dots-s_n} ds_1 \dots ds_n.$$

By Theorem 2.2 in [12], if there exists $m \in \mathbb{N}$ such that the remainder term $R_m(t)$ is weakly compact for each $t > 0$ (i.e. it maps any bounded set to relatively compact set in $L^1(\mathbb{T}^d \times V)$-weak), then for all $t > 0$

$$r_{\mathrm{ess}}(T_t) \leq r(S_t).$$

Therefore, establishing Proposition 6.2 reduces to proving that $R_m(t)$ is a weakly compact operator on $L^1(\mathbb{T}^d \times V)$ for all $t \geq 0$.

We recall moreover that $(A_t)_{t\geq 0}$ designates the semigroup associated with the free transport equation. In other words, for each $f \in L^1(\mathbb{T}^d \times V)$

$$(A_t f)(x,v) = f(x - tv, v).$$

Notice that for each $t > 0$,

$$0 \leq S_t \leq A_t,$$

where the order is taken in the sense of Remark 3.1. In the same way,

$$K_\sigma \leq \|\sigma k\|_{L^\infty(\mathbb{T}^d)} \tilde{M},$$

where $\tilde{M}$ designates the operator

$$\tilde{M}(f) := \int_V f(\,\cdot\,, v) dv,$$

for each $f \in L^1(\mathbb{T}^d \times V)$. Hence, for each $n \geq 1$,

$$0 \leq R_n(t) \leq \left(\|\sigma k\|_{L^\infty(\mathbb{T}^d)}\right)^n V_n(t) \tag{6.4}$$

with

$$V_n(t) := \int_{\substack{s_1+\cdots+s_n\leq t;\\ 0\leq s_i}} A_{s_1}\tilde{M}\ldots\tilde{M}A_{s_n}\tilde{M}T_{t-s_1-\cdots-s_n} ds_1\ldots ds_n.$$

Assume that there exists m such that $V_m(t)$ is weakly compact for each $t > 0$. Since $V_m(t)$ dominates $R_m(t)$ for each $t > 0$, that implies, by Proposition 2.1. in [7], that $R_m(t)$ is weakly compact for each $t > 0$. Therefore, establishing that the operator $V_m(t)$ is weakly compact on $L^1(\mathbb{T}^d \times V)$ for each $t > 0$ will prove that $R_m(t)$ is weakly compact on $L^1(\mathbb{T}^d \times V)$ for each $t > 0$. As a result, Proposition 6.2 will follow.

In the sequel, for the sake of simplicity, we will use the notation

$$V_n(t) = \int_0^t \left[A\tilde{M}\right]^n (s) T_{t-s} ds,$$

with

$$\left[A\tilde{M}\right](t) := A_t\tilde{M},$$

and

$$\left[A\tilde{M}\right]^k := \left[A\tilde{M}\right] * \left[A\tilde{M}\right] * \cdots * \left[A\tilde{M}\right] \ (k \text{ times}, k > 1),$$

where $*$ designates the convolution product in the time variable.

We then recall a proposition, which is a special case of a result established by M. Mokhtar-Kharroubi in [8].

Proposition 6.3. (Mokhtar-Kharroubi). *With the notations above, for each $m \geq d+1$ and for each $t > 0$ the operator $\left[A\tilde{M}\right]^m (t)$ is weakly compact on $L^1(\mathbb{T}^d \times V)$.*

With Proposition 6.3, we arrive at the following

Lemma 6.4. *Under the assumptions above, there exists $m \in \mathbb{N}$ such that $V_m(t)$ is weakly compact on $L^1(\mathbb{T}^d \times V)$ for each $t > 0$.*

Proof. Since the set of weakly compact operators, denoted here by $W(L^1(\mathbb{T}^d \times V))$, is an ideal for the Banach algebra $\mathcal{L}(L^1(\mathbb{T}^d \times V))$, Proposition 6.3 implies that, for each $t > 0$ and $0 < s < t$, the operator $\left[A\tilde{M}\right]^m (s) T_{t-s}$ is weakly compact. In other words, for each $t > 0$, the function $s \in [0,t] \mapsto \left[A\tilde{M}\right]^m (s) T_{t-s}$ is a $W(L^1(\mathbb{T}^d \times V))$-valued function on $[0,t]$.

Moreover, the ideal $W\left(L^1\left(\mathbb{T}^d\times V\right)\right)$ has the *Strong Convex Compactness Property*. This means that, if a $W\left(L^1\left(\mathbb{T}^d\times V\right)\right)$-valued function F defined on $[0,T]$ satisfies the following conditions

(1) $\sup_{t\in[0,T]}\|F\|_{\mathcal{L}(L^1(\mathbb{T}^d\times V))}<\infty$
(2) $\forall x\in L^1\left(\mathbb{T}^d\times V\right)$, the function $t\mapsto F(t)x$ is measurable

then

$$\int_0^T F(s)ds\in W\left(L^1\left(\mathbb{T}^d\times V\right)\right).$$

The second condition is immediate. In order to prove the first condition, we recall that for each $t\geq 0$

$$\left\|\tilde{M}\right\|_{\mathcal{L}(L^1(\mathbb{T}^d\times V))}=\|A_t\|_{\mathcal{L}(L^1(\mathbb{T}^d\times V))}=\|T_t\|_{\mathcal{L}(L^1(\mathbb{T}^d\times V))}=1,$$

so that, for each $t>0$,

$$\sup_{0<s<t}\left\|\left[A\tilde{M}\right]^m(s)T_{t-s}\right\|_{\mathcal{L}(L^1(\mathbb{T}^d\times V))}\leq 1.$$

Consequently, by Theorem 2.2 in [10], the Strong Convex Compactness Property implies that for each $t>0$

$$\int_0^t\left[A\tilde{M}\right]^m(s)T_{t-s}ds\in W\left(L^1\left(\mathbb{T}^d\times V\right)\right)$$

and hence, for each $t>0$, the operator $V_m(t)$ is weakly compact. □

As explained above, Proposition 6.2 is a straightforward consequence of Lemma 6.4.

6.3. **The asymptotic behavior of the essential radius.** In view of (6.1) and Proposition 6.2, in order to prove that $\mathcal{T}$ is quasi-compact on $L^1\left(\mathbb{T}^d\times V\right)$, it is enough to prove that for some $t_0>0$, $r\left(S_{t_0}\right)<1$. This is implied by the next proposition.

Proposition 6.5. *If σ verifies the geometrical condition, then*

$$\lim_{t\to+\infty}r\left(S_t\right)=0.$$

Proof. We recall that the geometrical condition means that there exist T_0 and C such that

$$\int_0^{T_0}\sigma(x-sv)ds>C\text{ a.e. in }(x,v)\in\mathbb{T}^d\times V.$$

Since $\sigma\geq 0$ we have, for each $t>T_0$,

$$\begin{aligned}\int_0^t\sigma(x-sv)ds&\geq\int_0^{\left\lfloor\frac{t}{T_0}\right\rfloor T_0}\sigma(x-sv)ds\\&\geq\sum_{n=0}^{\left\lfloor\frac{t}{T_0}\right\rfloor}\int_0^{T_0}\sigma\left((x-nT_0v)-sv\right)ds\\&\geq\left\lfloor\frac{t}{T_0}\right\rfloor C,\end{aligned}\tag{6.5}$$

(where $\lfloor x\rfloor$ designates the largest integer $\leq x$). Coming back to (6.2) we have, for all $g\in L^1\left(\mathbb{T}^d\times V\right)$ and $t\geq 0$,

$$|S_tg|\leq|g(x-vt,v)|\,e^{-\int_0^t\sigma(x-sv)ds}.$$

By (6.5), the inequality above entails that, for all $g \in L^1\left(\mathbb{T}^d \times V\right)$ and $t \geq T_0$,

$$|S_t g| \leq |g(x - vt, v)|\, e^{-C\left\lfloor \frac{t}{T_0} \right\rfloor}.$$

By integrating both sides of the inequality above in $x \in \mathbb{T}^d$ and in $v \in V$, we obtain

$$\|S_t g\|_{L^1(\mathbb{T}^d \times V)} \leq e^{-\left\lfloor \frac{t}{T_0} \right\rfloor C} \|g\|_{L^1(\mathbb{T}^d \times V)} \text{ for each } t \geq T_0,$$

and thus

$$\|S_t\|_{\mathcal{L}(L^1(\mathbb{T}^d \times V))} \leq e^{-\left\lfloor \frac{t}{T_0} \right\rfloor C} \text{ for each } t \geq T_0. \tag{6.6}$$

Since

$$r\left(S_t\right) \leq \|S_t\|_{\mathcal{L}(L^1(\mathbb{T}^d \times V))}$$

we have, by inequality (6.6),

$$r\left(S_t\right) \leq e^{-C\left\lfloor \frac{t}{T_0} \right\rfloor} \text{ for each } t \geq T_0. \tag{6.7}$$

This implies that

$$\lim_{t \to +\infty} r\left(S_t\right) = 0.$$

□

Summarizing the discussion above, we have proved that

Proposition 6.6. *If σ verifies the geometrical condition* (2.4)*, then the semigroup $\mathcal{T}$ is quasi compact on $L^1\left(\mathbb{T}^d \times V\right)$.*

7. The characterization of P

So far we have proved that $\mathcal{T}$ is a bounded, quasi-compact, irreducible, positive C_0-semigroup on $L^1\left(\mathbb{T}^d \times \ V\right)$ with spectral bound zero. Thus, by Theorem 3.6, there exist a positive, rank-one projection P and suitable constants $C \geq 1$ and $a > 0$ such that

$$\|T_t - P\|_{\mathcal{L}(L^1(\mathbb{T}^d \times \ V))} \leq Ce^{-at} \text{ for each } t \geq 0. \tag{7.1}$$

To conclude the proof of (2.5) in Theorem 2.2, it remains to identify the projector P in (7.1).

We recall that we have proved that if σ verifies the geometrical condition (2.4), then

$$\lim_{t \to +\infty} r_{\text{ess}}\left(T_t\right) = 0.$$

That implies by Theorem 3.1, chapter V in [6] that the spectrum of B is discrete and in particular that $s(A)$ is a pole of the resolvent $R(A)$. Besides, B is the generator of an irreducible semigroup $\mathcal{T}$ so that, by Proposition 3.5 C-III in [1], the residue P associated to $s(A) = 0$ is a projection onto Ker$B$, that is one-dimensional. Moreover $P = e \otimes \psi$ with $e \in \text{Ker}B$ and $\psi \in L^\infty\left(\mathbb{T}^d \times V\right)$ with $\psi \geq 0$.

We know that $1 \in \text{Ker}B$, which is one-dimensional, so that we can take $e = 1$. Notice that, by conservation of the mass, we have, for each $f \in L^1\left(\mathbb{T}^d \times V\right)$,

$$\int_{\mathbb{T}^d \times V} Pf(x, v)\, dxdv = \int_{\mathbb{T}^d \times V} f(x, v)\, dxdv.$$

Keeping it in mind, we notice that, since $\psi \in L^\infty\left(\mathbb{T}^d \times V\right)$, we have the following equality:

$$\int_{\mathbb{T}^d \times V} \psi^2\, dxdv = \int_{\mathbb{T}^d \times V} P\psi\, dxdv = \int_{\mathbb{T}^d \times V} \psi\, dxdv. \tag{7.2}$$

Besides, since $P = 1 \otimes \psi$ we have

$$\int_{\mathbb{T}^d \times V} \psi P f \, dxdv = \int_{\mathbb{T}^d \times V} \psi \, dxdv \int_{\mathbb{T}^d \times V} \psi f \, dxdv.$$

Taking $f = 1$, we obtain

$$\int_{\mathbb{T}^d \times V} \psi \, dxdv = \left(\int_{\mathbb{T}^d \times V} \psi \, dxdv \right)^2. \tag{7.3}$$

Hence by (7.2) and (7.3) we have the equality

$$\left(\int_{\mathbb{T}^d \times V} \psi \, dxdv \right)^2 = \int_{\mathbb{T}^d \times V} \psi^2 \, dxdv.$$

Since $x \mapsto x^2$ is strictly convex, Jensen's equality implies that ψ is a constant. Since $\int_{\mathbb{T}^d \times V} \psi = 1 \, dxdv$, we have

$$\psi = 1.$$

We have hence established that the geometrical condition implies the exponential decay estimate of Theorem 2.2.

8. On the sharpness of the geometrical condition

We conclude here the argument leading to the proof of Theorem 2.2 by showing that the geometrical condition (2.4) is also necessary to obtain exponential convergence to equilibrium for the solution of the Cauchy problem (2.1).

Notice that the previous sections show that all the conditions required in Theorem 3.6, except the quasi-compactness, are established without resorting to the geometrical condition.

However, the quasi-compactness is the critical point in the use of Theorem 3.6 to prove the exponential decay to equilibrium. Reciprocally, if the semigroup $\mathcal{T}$ converges uniformly to its equilibrium, then it is necessarily quasi-compact. Therefore, any obstruction to the exponential convergence can come only from obstructions to the quasi-compactness of $\mathcal{T}$.

Our approach is similar to Section 6: we first show that the quasi-compactness of $\mathcal{T}$ is equivalent to the quasi-compactness of $\mathcal{S}$, easier to study since we have an explicit formula for $\mathcal{S}$. Then, we show that the quasi-compactness of $\mathcal{S}$ and Birkhoff's ergodic theorem imply that S_t converges uniformly to zero as t tends to infinity. Finally, we prove that the geometrical condition is a necessary condition for obtaining the exponential convergence to zero for $\mathcal{S}$.

In other words, if we suppose that $\mathcal{T}$ converges to equilibrium at exponential convergence rate, then the geometrical condition holds.

8.1. **The quasi-compactness of $\mathcal{T}$ and $\mathcal{S}$.** The quasi-compactness of $\mathcal{T}$ in $L^1\left(\mathbb{T}^d \times V\right)$ implies the quasi-compactness of $\mathcal{S}$ in $L^1\left(\mathbb{T}^d \times V\right)$, as a consequence of Caselles' Theorem [4]:

Proposition 8.1. (Caselles). *Let E be a Banach lattice. Let $S, T \in \mathcal{L}(E)$ be such that*

$$0 \leq S \leq T.$$

If $r(T) \leq 1$ and $r_{\mathrm{ess}}(T) < 1$, then $r_{\mathrm{ess}}(S) < 1$.

Keeping this in mind, we can prove the following result:

Lemma 8.2. *The semigroup $\mathcal{S}$ is quasi-compact on $L^1\left(\mathbb{T}^d \times V\right)$ if $\mathcal{T}$ is quasi-compact on $L^1\left(\mathbb{T}^d \times V\right)$.*

Proof. We recall that, by Duhamel's Formula, we have the following equality:

$$T_t = S_t + \int_0^t S_s K_\sigma T_{t-s} ds, \text{ for all } t \geq 0.$$

Since $\mathcal{T}$ and $\mathcal{S}$ are positive semigroups and K_σ is a positive operator, we have

$$\int_0^t S_s K_\sigma T_{t-s} ds \geq 0 \text{ for each } t \geq 0.$$

Therefore the equality above implies that T_t dominates S_t for each $t \geq 0$:

$$T_t \geq S_t \text{ for each } t \geq 0. \tag{8.1}$$

Besides, we have shown that

$$r(T_t) = 1 \text{ for each } t \geq 0. \tag{8.2}$$

Since $\mathcal{T}$ is quasi-compact on $L^1\left(\mathbb{T}^d \times V\right)$, there exists t_0 such that

$$r_{\text{ess}}(T_{t_0}) < 1.$$

Hence Caselles' Theorem implies that

$$r_{\text{ess}}(S_{t_0}) < 1,$$

so that the semigroup $\mathcal{S}$ is quasi-compact on $L^1\left(\mathbb{T}^d \times V\right)$. □

8.2. **The asymptotic behavior of $\mathcal{S}$ in the long time limit.** Here we show that the quasi-compactness of $\mathcal{S}$, together with Birkhoff's ergodic theorem, implies convergence to zero in the long time limit. We begin with the following lemma:

Lemma 8.3. *If $\int_{\mathbb{T}^d} \sigma(x)dx > 0$ then*

$$\forall f \in L^\infty\left(\mathbb{T}^d \times V\right),\ S_t f \to 0 \textit{ in } L^1\left(\mathbb{T}^d \times V\right)\textit{-strong as } t \to +\infty.$$

Proof. Since the linear flow $x \mapsto \phi_{x,v}(t)$ is ergodic on $\mathbb{T}^d$ for a.e. $v \in V$ we deduce from Birkhoff's Theorem that, for a.e. $(x,v) \in \mathbb{T}^d \times V$,

$$\left| \frac{1}{T} \int_0^T \sigma(x - vs)ds - \int_{\mathbb{T}^d} \sigma(x)dx \right| \to 0 \text{ as } T \to +\infty.$$

Since by assumption

$$\int_{\mathbb{T}^d} \sigma(x)dx > 0,$$

the ergodic theorem implies in fact that

$$\int_0^T \sigma(x - vs)ds \to +\infty \text{ a.e. on } \mathbb{T}^d \times V$$

as $T \to +\infty$. Thus for each $f \in L^\infty\left(\mathbb{T}^d \times V\right)$,

$$\left| f(x - tv)e^{-\int_0^t \sigma(x-vs)ds} \right| \to 0 \text{ a.e. on } \mathbb{T}^d \times V \text{ as } t \to +\infty.$$

Since

$$\left| f(x - tv)e^{-\int_0^t \sigma(x-vs)ds} \right| \leq \|f\|_{L^\infty(\mathbb{T}^d \times V)}$$

with $(x,v) \mapsto \|f\|_{L^\infty(\mathbb{T}^d \times V)} \mathbb{1}_{\mathbb{T}^d \times V} \in L^1\left(\mathbb{T}^d \times V\right)$, we deduce by dominated convergence that, for each $f \in L^\infty\left(\mathbb{T}^d \times V\right)$

$$\|S_t f\|_{L^1(\mathbb{T}^d \times V)} \to 0 \text{ as } t \to +\infty.$$

□

Lemma 8.4. *If $\mathcal{S}$ is quasi-compact on $L^1\left(\mathbb{T}^d \times V\right)$ and if $\|S_t f\|_{L^1(\mathbb{T}^d\times V)} \to 0$ as $t \to +\infty$ for each $f \in L^\infty\left(\mathbb{T}^d \times V\right)$, then*

$$\|S_t\|_{\mathcal{L}(L^1(\mathbb{T}^d\times V))} \to 0 \text{ as } t \to +\infty.$$

Proof. That $\mathcal{S}$ is quasi-compact on $L^1\left(\mathbb{T}^d \times V\right)$ means that there exists a compact operator C on $L^1\left(\mathbb{T}^d \times V\right)$ such that

$$\|S_t - C\|_{\mathcal{L}(L^1(\mathbb{T}^d\times V))} \to 0 \text{ as } t \to +\infty.$$

Keeping this in mind, we know that, for $f \in L^\infty\left(\mathbb{T}^d \times V\right)$,

$$\|S_t f\|_{L^1(\mathbb{T}^d\times V)} \to 0 \text{ as } t \to +\infty.$$

By the density of $L^\infty\left(\mathbb{T}^d \times V\right)$ in $L^1\left(\mathbb{T}^d \times V\right)$, we conclude that $C \equiv 0$. □

Since $\int_{\mathbb{T}^d} \sigma(x)dx > 0$, assuming that $\mathcal{S}$ is quasi-compact in $L^1\left(\mathbb{T}^d \times V\right)$ implies that S_t converges to 0 in the operator norm topology.

8.3. **The geometrical condition.** We finally show that this asymptotic behaviour implies the geometrical condition (2.4).

Lemma 8.5. *If*

$$\|S_t\|_{\mathcal{L}(L^1(\mathbb{T}^d\times V))} \to 0 \text{ as } t \to +\infty,$$

then σ must satisfy the geometrical condition.

Proof. We assume that

$$\|S_t\|_{\mathcal{L}(L^1(\mathbb{T}^d\times V))} \to 0 \text{ as } t \to +\infty.$$

This means that, for each $\varepsilon > 0$ there exists t_0 such that, for each $t \geq t_0$,

$$\sup_{\|f\|_{L^1(\mathbb{T}^d\times V)}\leq 1} \|S_t f\|_{L^1(\mathbb{T}^d\times V)} \leq \varepsilon. \tag{8.3}$$

Assume that $\sigma \in C\left(\mathbb{T}^d \times V\right)$ and take, for each $(x, v) \in \mathbb{T}^d \times V$, a sequence of positive functions $(f_n)_{n\geq 1}$ such that

$$\int_{\mathbb{T}^d\times V} f_n(x, v)dxdv = 1 \text{ for all } n \in \mathbb{N}$$

with

$$f_n \to \delta_{x,v} \text{ as } n \to +\infty$$

in the sense of distributions. Then we have

$$\begin{aligned}
\|S_t f_n\|_{L^1(\mathbb{T}^d\times V)} &= \int_{\mathbb{T}^d\times V} \left(S_t f_n\right)(z, w)dzdw \\
&= \int_{\mathbb{T}^d\times V} f_n(z - vt)e^{-\int_0^t \sigma(z-ws)ds}dzdw \\
&= \int_{\mathbb{T}^d\times V} f_n(y)e^{-\int_0^t \sigma(y-w(t-s))ds}dydw \\
&\to e^{-\int_0^t \sigma(x-v(t-s))ds} \text{ as } n \to +\infty.
\end{aligned}$$

Hence, (8.3) implies that for each $t \geq t_0$

$$e^{-\int_0^t \sigma(x-v(t-s))ds} \leq \varepsilon$$

or, equivalently, for each $t \geq t_0$

$$\int_0^t \sigma(x - v(t - s)) \geq \ln\left(\frac{1}{\varepsilon}\right).$$

In other words, the geometrical condition holds.

□

9. Conclusion

In this article we have proved that the solution of the linear transport equation converges in the long time limit to its global equilibrium state at an exponential rate if and only if the cross section satisfies the geometrical condition (2.4).

This geometrical condition (2.4) can obviously be satisfied by cross sections vanishing on sets of positive measure (for instance, it is satisfied if the cross section is larger than a positive constant on the complement of a ball with radius $< 1/2$ in $\mathbb{T}^d$). In other words, exponential convergence to equilibrium depends on both the region where the cross section vanishes and the geometry of characteristic lines of the free transport operator.

Since our method of proof is based on compactness arguments, it is nonconstructive. Therefore the question of relating the best decay rate in (2.5) to the constants appearing in the geometrical condition (2.4) remains an open problem.

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