Exponential relaxation to self-similarity for the superquadratic fragmentation equation

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Abstract

We consider the self-similar fragmentation equation with a superquadratic fragmentation rate and provide a quantitative estimate of the spectral gap.

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1 Introduction

The *fragmentation equation*

$$\begin{cases} \partial_t f(t,x) = \mathcal{F}f(t,x), & t \geq 0,\ x > 0, \\ f(0,x) = f_{\rm in}(x), & x > 0 \end{cases} \tag{1}$$

is a model that describes the time evolution of a population structured with respect to the size x of the individuals.

The key term of the model is the fragmentation operator $\mathcal{F}$, defined as

$$\mathcal{F}f(x) := \int_x^\infty b(y,x)f(y)\,\mathrm{d}y - B(x)f(x). \tag{2}$$

The fragmentation operator quantifies the generation of smaller individuals from a member of the population of size x: the individuals split with a rate $B(x)$ and generate smaller individuals of size $y \in (0,x)$, whose distribution is governed by the kernel $b(x,y)$.

Models involving the fragmentation operator appear in various applications. Among them we can mention crushing of rocks, droplet breakup or combustion [2] which are pure fragmentation phenomena, but also cell division [14], protein polymerization [9] or data transmission protocols on the web [3], for which the fragmentation process occurs together with some "growth" phenomenon.

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In order to ensure the conservation of the total mass of particles which may occur during the fragmentation process, the coefficients $B(x)$ and $b(y,x)$ must be linked through the relation

$$\int_0^y xb(y,x)\,\mathrm{d}x = yB(y). \tag{3}$$

This assumption ensures, at least formally, the mass conservation law

$$\forall t>0, \qquad \int_0^\infty xf(t,x)\,\mathrm{d}x = \int_0^\infty xf_{\mathrm{in}}(x)\,\mathrm{d}x := \rho_{\mathrm{in}}. \tag{4}$$

Moreover, it is well known that $xf(t,x)$ converges to a Dirac mass at $x=0$ when $t\to+\infty$. Usually, the various contributions that are available in the literature restrict their attention to coefficients which satisfy the homogeneous assumptions (see [8] for instance)

$$B(x)=x^\gamma, \quad \gamma>0, \qquad \text{and} \qquad b(y,x)=y^{\gamma-1}p\Big(\frac{x}{y}\Big), \tag{5}$$

where $\mathrm{d}\mu(z):=p(z)\,\mathrm{d}z$ is a positive measure supported on $[0,1]$ which satisfies

$$\int_0^1 z\,\mathrm{d}\mu(z)=1.$$

These hypotheses guarantee that the relation (3) is verified.

From a mathematical point of view, it is convenient to perform the (mass preserving) self-similar change of variable

$$f(t,x)=(1+t)^{2/\gamma}g\Big(\frac{1}{\gamma}\log(1+t),(1+t)^{1/\gamma}x\Big),$$

or, by writing g in terms of f,

$$g(t,x):=e^{-2t}f(e^{\gamma t}-1,e^{-t}x).$$

It allows to deduce that $g(t,x)$ satisfies the *self-similar fragmentation equation*

$$\begin{cases} \partial_t g+\partial_x(xg)+g=\gamma\mathcal{F}g, & t\geq 0,\ x>0,\\ g(0,x)=f_{\mathrm{in}}(x), & x>0. \end{cases} \tag{6}$$

Equation (6) belongs to the class of *growth-fragmentation equations* and it admits – unlike Equation (1) – positive steady-states [7, 8, 15].

Denote by G the unique positive steady-state of Equation (6) with normalized mass, i.e. the solution of

$$(xG)'+G=\gamma\mathcal{F}G, \qquad G>0, \qquad \int_0^\infty xG(x)\,\mathrm{d}x=1.$$

Then it has been proved (see [8, 12]) that the solution $g(t,x)$ of the self-similar fragmentation equation (6) converges to $\rho_{\mathrm{in}}G(x)$ when $t\to+\infty$.

Coming back to the fragmentation equation (1), this result implies the convergence of $f(t,x)$ to the self-similar solution $(t,x)\mapsto\rho_{\mathrm{in}}(1+t)^{2/\gamma}G((1+t)^{1/\gamma}x)$ and hence the convergence of $xf(t,x)$ to a Dirac mass δ_0.

In order to obtain more precise quantitative properties of the previous equation, one can wonder about the rate of convergence of $g(t,x)$ to the asymptotic profile $G(x)$. Many recent works are dedicated to this question and prove, under different assumptions and with different techniques, an exponential rate of convergence for growth-fragmentation equations [1, 3, 4, 5, 6, 11, 13, 15].

Nevertheless, to our knowledge the only results about the specific case of the self-similar fragmentation equation are those provided by Cáceres, Cañizo and Mischler [4, 5]. They prove exponential convergence in the Hilbert space $L^2((x+x^k)\,\mathrm{d}x)$ for a sufficiently large exponent k in [5], and in the Banach space $L^1((x^m+x^M)\,\mathrm{d}x)$ for suitable exponents $1/2<m<1<M<2$ in [4]. For proving their results, the authors of the aforementioned articles require the measure p to be a bounded function (from above and below) and the power γ of the fragmentation rate to be less than 2.

The current paper aims to obtain a convergence result for super-quadratic rates, namely when $\gamma \geq 2$. We obtain exponential convergence to the asymptotic state by working in the weighted Hilbert space $L^2(x\,\mathrm{d}x)$, under the following assumptions:

$$\gamma \geq 2 \qquad \text{and} \qquad p(z) \equiv 2. \tag{7}$$

The fact that $p(z)$ is a constant means that the distribution of the fragments is uniform: the probability to get a fragment of size x or x' from a particle of size $y > x, x'$ is exactly the same. Then the condition $\int_0^1 zp(z)\,\mathrm{d}z = 1$ imposes this constant to be equal to 2, meaning that the fragmentation is necessarily binary. Our assumption on p is more restrictive than in [4, 5], but in return we get a stronger result in the sense that we obtain an estimate of the exponential rate. Now we state the main theorem of this paper.

Theorem 1.1. *Let $g_{\mathrm{in}} \in L^1(x\,\mathrm{d}x) \cap L^2(x\,\mathrm{d}x)$ and let $g \in C([0,\infty), L^1(x\,\mathrm{d}x))$ be the unique solution of the self-similar fragmentation equation* (6) *with initial condition g_{in} and with fragmentation coefficients satisfying* (5) *and* (7)*, that is*

$$B(x) = x^\gamma, \quad \gamma \geq 2 \qquad \textit{and} \qquad b(y,x) = 2y^{\gamma-1}.$$

Then the following estimate holds:

$$\|g(t,\cdot) - \rho_{\mathrm{in}} G\|_{L^2(x\,\mathrm{d}x)} \leq \|g_{\mathrm{in}} - \rho_{\mathrm{in}} G\|_{L^2(x\,\mathrm{d}x)}\, e^{-t}, \qquad t \geq 0.$$

2 Preliminaries

Define the suitable weighted spaces

$$\dot{L}^p_k := L^p(\mathbb{R}^+, x^k\ \mathrm{d}x) \quad \text{for } p \geq 1,\ k \in \mathbb{R}, \qquad \text{and} \qquad \dot{W}^{1,1}_1 := W^{1,1}(\mathbb{R}^+, x\ \mathrm{d}x).$$

For $u \in \dot{L}^1_1$ we denote moreover by

$$M(x) := \int_0^x yu(y)\ \mathrm{d}y$$

the primitive of $xu(x)$ which vanishes at $x=0$.

Now we recall the following existence and uniqueness result of a solution to the fragmentation equation, easily deduced from [8], Theorems 3.1-3.2 and Lemma 3.4:

Theorem [8]. *If*

$$B(x) = x^\gamma, \quad \gamma \ge 2 \qquad \text{and} \qquad b(y,x) = 2y^{\gamma-1},$$

for any $f_{\rm in} \in \dot L^1_m \cap \dot L^1_1$ with $m < 1$, there exists a unique solution $f \in C([0,\infty); \dot L^1_1)$ to the fragmentation equation (1) *such that the mass conservation* (4) *is satisfied. If, moreover, $f_{\rm in} \in \Xi := L^1 \cap \dot L^1_\gamma \cap \dot W^{1,1}_1$, the associated solution g to the self-similar fragmentation equation is such that*

$$(g(t,\cdot))_{t\ge0} \text{ is uniformly bounded in } \Xi.$$

In the following lemma we give some useful properties of the set

$$\Xi = L^1 \cap \dot L^1_\gamma \cap \dot W^{1,1}_1$$

and of the subset

$$\Xi_0 := \left\{ u \in \Xi, \ \int_0^\infty x u(x)\ {\rm d}x = 0 \right\}.$$

Lemma 2.1. *The set $\Xi = L^1 \cap \dot L^1_\gamma \cap \dot W^{1,1}_1$ satisfies*

$$G \in \Xi, \qquad \Xi \subset C(0,\infty) \cap \dot L^2_1 \cap \dot L^2_{\gamma+1} \qquad \text{and}$$

$$\forall u \in \Xi, \qquad \lim_{x\to0} x u(x) = \lim_{x\to+\infty} x u(x) = 0.$$

Moreover, if $\gamma \ge 1$, for any function $u \in \Xi_0$ the following inequality holds:

$$\forall x > 0, \qquad |M(x)| \le x^{1-\gamma} \|u\|_{\dot L^1_\gamma}.$$

Proof. The fact that the steady-state G belongs to Ξ is a consequence of the estimates in [1]. In the case when $p(z) \equiv 2$ it can also be deduced from the explicit formula (see [7])

$$G(x) = \frac{\gamma^{1-2/\gamma}}{\Gamma(2/\gamma)}\, e^{-\frac{x^\gamma}{\gamma}}$$

where Γ is the Euler Gamma function.

For $\gamma \ge 1$ and $u \in \Xi$ such that $\int_0^\infty x u(x)\ {\rm d}x = 0$ we can write for $x > 0$

$$|M(x)| = \left| - \int_x^\infty y u(y)\ {\rm d}y \right| \le x^{1-\gamma} \int_0^\infty y^\gamma |u(y)|\ {\rm d}y.$$

□

3 Proof of the main theorem

Define the self-similar fragmentation operator $\mathcal{L}u := -(xu)' - u + \gamma \mathcal{F} u$ and denote by

$$(u,v) := \int_0^\infty x u(x) v(x)\ {\rm d}x$$

the canonical scalar product in $\dot L^2_1$. Theorem 1.1 is a consequence of the following result.

Theorem 3.1. *Under Assumptions* (5) *and* (7), *i.e. for*

$$B(x) = x^\gamma, \quad \gamma \geq 2 \qquad \textit{and} \qquad b(y,x) = 2y^{\gamma-1},$$

we have

$$\forall u \in \Xi_0, \qquad (u, \mathcal{L}u) \leq -\|u\|^2_{\dot{L}^2_1}.$$

Proof. Using Lemma 2.1 we can deduce, for $u \in \Xi_0$,

$$(u,(xu)') = \int_0^\infty xu(x)\big(xu(x)\big)' \,\mathrm{d}x = \frac{1}{2}\int_0^\infty \big((xu(x))^2\big)' \,\mathrm{d}x = 0$$

and

$$\begin{aligned}
(u,\mathcal{F}u) &= 2\int_0^\infty xu(x)\int_x^\infty y^{\gamma-1}u(y)\,\mathrm{d}y\,\mathrm{d}x - \int_0^\infty x^{\gamma+1}u^2(x)\,\mathrm{d}x \\
&= 2\int_0^\infty x^{\gamma-1}u(x)\int_0^x yu(y)\,\mathrm{d}y\,\mathrm{d}x - \int_0^\infty x^{\gamma+1}u^2(x)\,\mathrm{d}x \\
&= 2\int_0^\infty x^{\gamma-2}M'(x)M(x)\,\mathrm{d}x - \int_0^\infty x^{\gamma+1}u^2(x)\,\mathrm{d}x \\
&= -(\gamma-2)\int_0^\infty x^{\gamma-3}M^2(x)\,\mathrm{d}x - \int_0^\infty x^{\gamma+1}u^2(x)\,\mathrm{d}x \leq 0.
\end{aligned}$$

□

Proof of Theorem 1.1. Assume first that $g_{\text{in}} \in \Xi$. From Theorem 3.1 we obtain the differential inequality

$$\frac{d}{dt}\|g(t,\cdot) - \rho_{\text{in}}G\|_{\dot{L}^2_1} \leq -\|g(t,\cdot) - \rho_{\text{in}}G\|_{\dot{L}^2_1}$$

which gives the result. Then we may remove the additional assumption $g_{\text{in}} \in \Xi$. □

4 Conclusion

We have proved a spectral gap result for the self-similar fragmentation operator $\mathcal{L}$ with a superquadratic fragmentation rate $B(x)$. More precisely we have obtained that this spectral gap is larger than 1. This is a new result concerning the long-time behaviour of the fragmentation equation (1). It also allows to extend the results obtained in [10] for non-linear growth-fragmentation equations.

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